BICOHERENT STATES, PATH INTEGRALS, AND SYSTEMS WITH CONSTRAINTS

By

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To the dawn of the day when man on earth, as in heaven, will be judged only by the nobility of his deeds and nothing else.

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In this work we discuss how dynamically constrained systems are handled in a path
integral framework. We consider models with a finite number of degrees of freedom
and with constraints that are either first class or second class. We first briefly review the
techniques developed in the past to handle such systems. Then we describe in detail the
method developed here which is summarized as follows: To account for the constraints
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the path integral representation of the propagator. The measure which is path dependent
is 'modulated' by the imaginary part of the action.

CHAPTER 1
INTRODUCTION

Modern physical theories of fundamental significance tend to be gauge theories and such theories have been the focus of extensive studies for the better part of this century. For instance, Quantum electrodynamics, a theory with an Abelian gauge group has been investigated for over fifty years now [1–4]. Also, Yang-Mills theories, models with non-Abelian gauge groups, have been at the center stage in the recent past [5–7]. It is a well known fact that dynamical systems with gauge symmetries have unphysical degrees of freedom. In the Hamiltonian formalism, their dynamics are described by constrained Hamiltonians. Hamiltonian systems with constraints are the main concern of the present thesis. In this brief study we shall limit ourselves to systems with a finite number of degrees of freedom so as to not lose sight of the main issues involved.

In describing physical theories, one usually starts with an action principle since it is easy to build Lagrangians with desired symmetries, such as relativistic invariance. Then, as a first step toward quantization one passes to the Hamiltonian formalism at which stage one promotes classical variables to quantum operators. It must be noted though that there exist methods of quantization where one passes directly from the Lagrangian to the quantum theory, hence bypassing the Hamiltonian formalism. But these schemes work only for Lagrangians quadratic in the velocities. Moreover, for their verification one has to go back to the Hamiltonian formalism. In the present study we shall take the Hamiltonian route to quantization.

We would now like to introduce some of the terminology used in the field of constraints. Constraints are classified depending on the manner in which they are expressed. If the constraints are expressed as equations connecting the Cartesian coordinates of the

system, they are Holonomic constraints. Nonholonomic constraints are either of two types: 1. They are expressed as inequalities for certain functions of the coordinates. 2. They are linear non-integrable relations among the differentials of these coordinates. An example of a system having Holonomic constraints is a rigid body where the constraints are given by equations of the form

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij} = 0. \tag{1.1}$$

In chapter 5 we study the (a, b, c) model which is a system with a Holonomic constraint. An example of a system with a nonholonomic constraint of type 1 is particles confined inside a box. Such constraints, which tend to appear in the macroscopic world, can be handled by introducing forces of constraints that prevent the inequality from being violated [8,9]. We are not interested in such systems here and shall not mention them anymore. Nonholonomic constraints of type 2 are called dynamical constraints and systems having such constraints will be the main focus of our study.

Dynamically constrained systems were systematically studied by Dirac and we now briefly review the analysis of such systems [10]. The starting point of our discussion will be the action integral

$$S = \int L(q, \dot{q}) dt. \tag{1.2}$$

The classical trajectories of the system are those that make the action stationary and are determined by the Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_n}) - \frac{\partial L}{\partial q_n} = 0, \qquad n = 1, 2, ...N. \tag{1.3}$$

Here q_n and $\dot{q}_n$ are the generalized coordinates and velocities respectively and N is the number of degrees of freedom. The equations of motion (3) are second order differential equations and the dynamical information contained in each of them can

be equivalently stated with two independent first order equations. Dynamics in the Hamiltonian formulation is described by first order equations. To go over to the Hamiltonian formalism, we first introduce the canonically conjugate momentum variable defined by

$$p_n = \frac{\partial L}{\partial \dot{q}_n}. \tag{1.4}$$

The p's as defined, above and the q's, are the dynamical variables of the Hamiltonian formalism. From the momenta defined in (4) one can construct a matrix called the Hessian, which is defined below. One finds that if the determinant of the Hessian vanishes, i.e. if

$$\det \left\| \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_m} \right\| = 0, \tag{1.5}$$

not all the p_n's are independent. The model then is said to have constraints, i.e. there exist certain relations of the type

$$\phi_m(q, p) = 0, \qquad m = 1, 2, ...M. \tag{1.6}$$

These relations are called *Primary Constraints*. The rank of the Hessian is equal to $N - M$ if the primary constraints (6) are all independent. The primary constraints, which are identities when the definition $p_n = (\partial L / \partial \dot{q}_n)$ is substituted into (6), must satisfy certain regularity conditions. These conditions can be stated as follows: In any variation for which δq and δp are $\mathcal{O}(\varepsilon)$ the constraints must be such that $\delta \phi_m(q, p)$ are also of $\mathcal{O}(\varepsilon)$. For example, of the following three ways of stating the same constraint,

$$p = 0, \qquad p^2 = 0, \qquad \sqrt[3]{p} = 0, \tag{1.7}$$

only the first one is acceptable [11,12].

We now introduce the canonical Hamiltonian, $H = p_n \dot{q}_n - L$. Here and in what follows, repeated indices are summed over unless otherwise stated. Notice that the

Hamiltonian defined in this way is not unique since we may add to it any linear combination of the constraints. Next, using $p_n = (\partial L/\partial \dot{q}_n)$ one can easily see that $\delta H = \dot{q}_n \delta p_n - (\partial L/\partial q_n)\delta q_n$. But the δq's and the δp's are not independent in this equation because of the constraint $\phi_m(q,p) = 0$ relating them. Now, from the general methods of the calculus of variations for systems with constraints one finds

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + u_m \frac{\partial \phi_m}{\partial p_n},$$
$$\frac{\partial L}{\partial q_n} = \dot{p}_n = -\frac{\partial H}{\partial q_n} - u_m \frac{\partial \phi_m}{\partial q_n}, \tag{1.8}$$

where the u_m's are undetermined coefficients. These equation are the *Hamilton's equations of motion* for a constrained system. These equations of motion can be written in a compact and elegant form using the *Poisson bracket*, which for two arbitrary functions $f(q,p)$ and $g(q,p)$ is defined by

$$\{f(q,p),g(q,p)\} = \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n}. \tag{1.9}$$

Thus, using the Poisson bracket notation we can write the time development of an arbitrary function $g(q,p)$ as

$$\dot{g}(q,p) = \frac{\partial g}{\partial q_n}\dot{q}_n + \frac{\partial g}{\partial p_n}\dot{p}_n,$$
$$= \{g,H\} + u_m\{g,\phi_m\}. \tag{1.10}$$

At this point we would like to introduce the idea of weak equality ($\approx$). We accept as a rule that all Poisson brackets must be worked out before the constraint equations are imposed [10]. To remind us of this rule we write the constraint equations using a slightly different equality sign as

$$\phi_m(q,p) \approx 0. \tag{1.11}$$

Such equations are called weak equations to distinguish them from the usual or 'strong equation.' We can now write the evolutions equations for an arbitrary dynamical variable

concisely, using the idea of weak equality, as

$$\dot{g}(q,p) \approx \{g, H_T\}. \tag{1.12}$$

In the expression above we have introduced, $H_T = H + u_m \phi_m$, the total Hamiltonian.

Let us now examine the consequences of the Hamilton equations of motion. We want the constraints to be zero at all times, i.e. we want

$$\dot{\phi}_m \approx \{\phi_m, H\} + u_n\{\phi_m, \phi_n\} = 0. \tag{1.13}$$

The above equation can lead to one of several possibilities: It can reduce to an equation, independent of the u's, involving only the dynamical variables of the form

$$\chi(q,p) = 0. \tag{1.14}$$

If this equation is independent of the primary constraints then we have a new constraint in the theory and such constraints are called *secondary constraints*. They are referred to as secondary because the equations of motion have been used to arrive at them. Continuing in the same vein, we require that the new constraint hold at all times and so

$$\dot{\chi} \approx \{\chi, H\} + u_m\{\chi, \phi_m\} = 0. \tag{1.15}$$

The above equation can lead to more constraints. In this manner we could obtain all constraints in the theory. We shall use the following notation to represent the secondary constraints

$$\phi_k(q,p) \approx 0, \qquad k = M+1, ..., M+K. \tag{1.16}$$

Next, in the case where equations (13) and (15) are not independent of the u_m's these are conditions on the coefficients u_m. Thus, the coefficient u_m's are such that they satisfy the equation

$$\{\phi_j, H\} + u_m\{\phi_j, \phi_m\} \approx 0. \tag{1.17}$$

In the expression above the ϕ_j's include both the primary and the secondary constraints, i.e. $j = 1, ..., M + K$, whereas the ϕ_m's denote only the primary constraints. Equation (17) are non-homogeneous linear equations in the unknowns u_m's and the general solution to these equation are

$$u_m = U_m(q, p) + v_a V_{am}(q, p), \qquad (1.18)$$

where $U_m(q, p)$ are particular solutions of (17) and $V_{am}(q, p)$ are solutions of the corresponding homogeneous equation $V_m\{\phi_j, \phi_m\} = 0$. The coefficients v_a are arbitrary and can be time dependent. So, the total Hamiltonian can now be written as

$$H_T = H + U_m\phi_m + v_a\phi_a, \qquad (1.19)$$

where $\phi_a = V_{am}\phi_m$, are linear combinations of only the primary constraints.

We would now like to introduce two new terms. A function, $f(q, p)$ is said to be *first-class* if its Poisson bracket is weakly zero with all the constraints $\phi_j(q, p)$, i.e.,

$$\{f(q, p), \phi_j(q, p)\} \approx 0. \qquad (1.20)$$

If the above is not true then $f(q, p)$ is *second-class*. It can be easily checked that our total Hamiltonian, $H_T = H + U_m\phi_m + v_a\phi_a$, is expressed as the sum of two first-class terms, $H + U_m\phi_m$ and a linear combination of the primary first-class constraints $v_a\phi_a$.

Let us now examine the transformations generated by the first-class constraints. Going back to our total Hamiltonian, recall that the coefficients v_a are totally arbitrary. So, for a general dynamical variable $f(q, p)$ with initial value f_0, its value at an infinitesimal time δt later is

$$f(\delta t) = f_0 + \delta t\{f, H + U_m\phi_m\} + \delta t v_a\{f, \phi_a\}, \qquad (1.21)$$

i.e. there is an arbitrary function of time in the evolution of $f(q, p)$ since it contains the coefficient v_a. Had we chosen a different coefficient such as v_a', we would have arrived

at a different value for $f(\delta t)$. The difference in the value of $f(\delta t)$ corresponding to the different coefficients, is

$$\Delta f(\delta t) = \delta t (v_a - v'_a)\{f, \phi_a\}. \tag{1.22}$$

Since v_a and v'_a are arbitrary coefficients, in our analysis, the two different values $f(\delta t)$ and $f'(\delta t)$ for the dynamical variable $f(q, p)$ must correspond to the same physical state. Thus, we conclude that first-class primary constraints are *generating functions of contact transformations that do not affect the physical state of the system. Such transformations are also called gauge transformations.*

From the discussion so far we have learned that the important classification of constraints is the one where they are classified as either first-class or second-class constraints. We also saw that the primary first-class constraints are generators of gauge transformations that do not alter the physical state of the system. Thus, from the standpoint of dynamics the primary and secondary constraints must be treated on equal footing and therefore one postulates that all first-class constraints, both primary and secondary, are generators of gauge transformations. Hence, we introduce an *Extended Hamiltonian* which includes all first-class constraints each accompanied by an arbitrary coefficient

$$H_E = H + U_m \phi_m + u_a \phi_a, \tag{1.23}$$

where the u_a's are arbitrary coefficients and can be functions of time. The extended Hamiltonian generates the most general time development which allows for arbitrary gauge transformation to be performed while the system dynamically evolves [12].

We now address the question of quantizing a constrained Hamiltonian theory. First consider the case where there are only first-class constraints present. We promote the dynamical variables, the p's and the q's, to operators. Then, we replace the

Poisson brackets of dynamical variables with commutation relations for the corresponding operators, i.e. $\{\ ,\ \} \rightarrow (1/i\hbar)[\ ,\ ]$. The Schrödinger equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi, \tag{1.24}$$

with the following supplementary conditions on the wave functions

$$\phi_j\psi = 0. \tag{1.25}$$

The wave functions that satisfy the above supplementary conditions are the physical states of the system. One finds that two additional conditions are needed for the quantization to be successful. Thus, for the supplementary conditions to be consistent we must have

$$[\phi_j, \phi_k]\psi = 0. \tag{1.26}$$

The commutator of the constraints acting on the wave functions will vanish if it can be written as $[\phi_j, \phi_k] = c_{jkl}\phi_l$. We know that in the classical theory the ϕ's are all first-class and so their Poisson brackets amongst themselves are linear combinations of some or all of the ϕ's, i.e. $\{\phi_j, \phi_k\} = c_{jkl}(q,p)\phi_l$. However, when we go over to the quantum theory the coefficients c's, which are functions of the coordinates and momenta, may not commute with the ϕ's and so the condition (26) need not be satisfied. If such ordering problems occur we might be able to achieve only a first approximation to the quantum theory, with quantities of order $\mathcal{O}(\hbar^2)$ neglected in trying to get the c's to be on the left of ϕ's. It turns out a second condition is required of the constraints as operators. The second condition arises as follows: As the wave functions which satisfy the supplementary condition (25) evolve in time according to the Schrödingers equation we want them to respect this condition at all times and this leads to the following requirements of the constraints

$$[\phi_j, H]\psi = 0. \tag{1.27}$$

The equation above implies that $[\phi_j, H] = b_{jk}\phi_k$. Again, in the classical theory, the coefficients b_{jk} are functions of the coordinates and momenta in general and so in going to the quantum theory we might encounter ordering problems and be able to obtain only a first approximation to the quantum theory. Thus, one sees that the quantization process may not be smooth and one might encounter quiet a few pitfalls on the way if one is able to quantize a given theory at all [10].

Let us now consider the case where second-class constraints are present. Recall that a constraint is second-class when its Poisson bracket with at least one of the other constraints is weakly nonvanishing. We shall denote the second-class constraints by $\chi_s(q, p)$. One finds that when the second-class constraints are such that no linear combination of them can be converted to a first-class constraint then the number of second-class constraints is even [10]. The simplest example of two second-class constraints is

$$q_1 \approx 0, \qquad p_1 \approx 0. \tag{1.28}$$

In the above example it is quite obvious that the degree of freedom labeled by the variables (q_1, p_1) is spurious and the theory must be formulated so as to eliminate this degree of freedom. To achieve this goal of effectively eliminating the spurious degrees of freedom in the case of theories with second-class constraints, Dirac constructed a new kind of Poisson bracket. These brackets have been referred to as the *Dirac brackets* in the literature since their introduction and are constructed as follows: Weakly evaluate the matrix $\{\chi_s, \chi_r\}$, i.e. evaluate the Poisson brackets and then set all the constraints equal to zero. It can be shown that the determinant of this matrix is nonvanishing and hence is invertible. Let the inverse of the matrix $\{\chi_s, \chi_r\}$ be C, i.e. $C_{rs}\{\chi_s, \chi_l\} \approx \delta_{rl}$. The Dirac bracket of two quantities $\xi(q, p)$ and $\eta(q, p)$ is defined as

$$\{\xi, \eta\}_{DB} \equiv \{\xi, \eta\} - \{\xi, \chi_s\}C_{sr}\{\chi_r, \eta\}. \tag{1.29}$$

The Dirac brackets satisfy all the identities satisfied by the Poisson brackets including the Jacobi identity [10]. Two things are noteworthy about the Dirac brackets: First, the equations of motion are as valid for the Dirac Brackets as for the original Poisson brackets, since for any $f(q,p)$

$$\{f, H_T\}_{DB} = \{f, H_T\} - \{f, \chi_s\} C_{sr} \{\chi_r, H_T\}$$
$$\approx \{f, H_T\}.$$

(1.30)

The above equation is true because $\{\chi_r, H_T\}$ weakly vanishes. Thus, we can write for the evolution of an arbitrary dynamical variable $\dot{f} \approx \{f, H_T\}_{DB}$. Second, the Dirac bracket of any function $\xi(q,p)$ of the coordinates and the momenta vanishes with all of the χ's since,

$$\{\xi, \chi_l\}_{DB} = \{\xi, \chi_l\} - \{\xi, \chi_r\} C_{rs} \{\chi_s, \chi_l\}$$
$$= \{\xi, \chi_l\} - \{\xi, \chi_r\} \delta_{rl} = 0.$$

(1.31)

Thus, if we use Dirac brackets in dynamical equations we can set

$$\chi_s(q,p) = 0$$

(1.32)

as strong equations. Hence, if we modify the classical theory by replacing the Poisson brackets with Dirac brackets, the passage to quantum theory is achieved by making the commutation relation correspond to the Dirac brackets. Then, equation (32) is taken to be an equation between the operators in the quantum theory. In practice, it must be noted, operator solutions to (32) might not be possible.

Thus far we have briefly introduced the language of constrained dynamics and described Dirac's prescription for operator quantization of such systems. Operator quantization is inelegant and not readily amenable to approximation schemes when one studies field theories. Hence, we would like to study the problem of handling theories with constraints in a path integral framework. The rest of this study is devoted to this problem. We have organized the thesis as follows: In chapter 2, we briefly review the existing

path integral techniques for handling systems with first and second-class constraints. In chapter 3, we describe a new technique for constructing path integral representations of canonical coherent-state propagators using *Bicoherent States* for systems with first-class constraints. Chapter 4 deals with the question of path integrals for systems with second-class constraints. Here we extend to such systems the techniques of chapter 3. In chapter 5, we construct the "Universal propagator" for a particle confined to a circular path. This is the simplest system with an holonomic constraint. We show in appendix B that this system can be considered at as a model with second-class constraints.

CHAPTER 2
CONSTRAINTS AND PATH INTEGRALS

In this chapter we briefly review the existing techniques for handling systems with constraints in a path integral framework. We will confine our attention to systems with a finite number of degrees of freedom as already noted. In section 2.1 we discuss Faddeev's scheme of quantization of systems with first-class constraints [13,14]. In section 2.2 we summarize the generalization of Faddeev's results to the case when second-class constraints are present. This generalization is due to Senjanovich [15]. In section 2.3 we review the methods of extended coherent states developed by Klauder and Whiting [16]. Extended coherent states introduce auxiliary variables which lead to a resolution of unity with a non-unique measure. These states, when used in the construction of path integral representations of propagators, lead to an action containing 'extra' path-space variables which engender constraints in the classical limit.

2.1 Systems with First-class Constraint

We now discuss Faddeev's quantization of systems with first-class constraints. Consider a mechanical system of N degrees of freedom described by the Lagrangian

$$L = L(q, \dot{q}). \tag{2.1}$$

If the above Lagrangian is to describe a constrained system, it must be singular, i.e., the determinant of its Hessian must vanish. We denote the constraints by

$$\phi_j(q, p) = 0, \qquad j = 1, 2, ..., J. \tag{2.2}$$

The constraints above include both primary and secondary constraints. We assume all the constraints are independent and first-class, i.e.

$$\{H, \phi_j\} = b_{jl}\phi_l, \qquad \{\phi_j, \phi_k\} = c_{jkl}\phi_l, \qquad j, k, l = 1, 2, ..., J. \tag{2.3}$$

Thus the constraint hypersurface M in the phase space Γ is a surface of dimension $2N - J$. It was noted in the introduction that all first-class constraints generate gauge transformations. Now, only those functions whose equations of motion do not contain any arbitrary functions of time on the constraint hypersurface M are defined as observables. Recall, the equation of motion of a arbitrary function $f(q,p)$ is

$$\frac{df}{dt} = \{f, H\} + v_j\{f, \phi_j\}, \tag{2.4}$$

where the coefficients v_j are functions of time. For $f(q,p)$ to be an observable, the second term on the right hand side of the above equation must vanish on the constraint surface, i.e., we require

$$\{f, \phi_j\} = d_{jk}\phi_k, \qquad j, k = 1, 2, ..., J. \tag{2.5}$$

The above equations can be viewed as a set of J first order differential equations on the manifold M with (3) serving as integrability conditions. Since $f(q,p)$ satisfies a set of J first order differential equations, it is completely determined by its values on a submanifold of dimension $2N - J - J = 2(N - J)$. It is convenient to take as such a manifold a surface $\Gamma^\star$, defined by the constraint equations (2) and J additional conditions

$$\chi_j(q,p) = 0, \qquad j = 1, 2, ..., J. \tag{2.6}$$

The above conditions are the gauge fixing conditions and they must be such that

$$\det \|\{\chi_j, \phi_k\}\| \neq 0. \tag{2.7}$$

This allows the manifold $\Gamma^\star$ to be an initial surface for the first order differential equations (5). It is also convenient to suppose that the gauge conditions mutually commute, i.e.,

$$\{\chi_j, \chi_k\} = 0. \tag{2.8}$$

If the gauge conditions meet the above requirements we can introduce a canonical transformation in Γ, such that in the new set of coordinates the gauge conditions take on the simple form

$$\chi_j(q,p) = P_j, \qquad j = 1, 2, ..., J. \tag{2.9}$$

The P_j's are a subset of the new canonical momentas. Equation (7) in the new coordinates becomes

$$\det \|\frac{\partial \phi_k}{\partial Q_j}\| \neq 0. \tag{2.10}$$

This equation can be solved for Q_j. Thus, the surface $\Gamma^\star$ is given by the equations

$$P_j = 0, \qquad Q_j = Q_j(q^\star, p^\star), \tag{2.11}$$

where the $q^\star$'s and $p^\star$'s are independent canonical variables on $\Gamma^\star$.

2.1.1 Propagator

We now study path integral representations of the propagator. The basic assertion is that the matrix element of the evolution operator is given by

$$\int e^{i \int_0^T [p_i \dot{q}_i - H(q,p)] dt} \prod_t d\mu(q(t)p(t)) \tag{2.12}$$

where the measure at each time slice in the integral above is given by

$$d\mu(t) = \frac{1}{(2\pi)^{N-J}} \det \|\{\chi_j, \phi_k\}\| \prod_j \delta(\chi_j)\delta(\phi_j) \prod_{i=1}^{N} dq_i(t)dp_i(t). \tag{2.13}$$

To prove this assertion we transform to the coordinates $(Q_j, q^\star, P_j, p^\star)$ in the integral (12). Thus, with these coordinates the measure in (12) becomes

$$d\mu(t) = \frac{1}{(2\pi)^{N-J}} \det \|\frac{\partial \phi_k}{\partial Q_j}\| \prod_j \delta(P_j)\delta(\phi_j) \prod_{i=1}^{N} dq_i(t)dp_i(t) \tag{2.14}$$

which can be rewritten as

$$d\mu(t) = \prod_{j=1}^{J} \delta(P_j)\delta(Q_j - Q_j(q^\star, p^\star))dQ_jdP_j \prod_{i=1}^{N-J} \frac{dq_i^\star dp_i^\star}{(2\pi)^{N-J}}. \qquad (2.15)$$

Now one can easily perform the (Q_j, P_j) integrals, thanks to the delta functions. Thus, the propagator now becomes

$$\int e^{i\int_0^T [\sum p_i^\star \dot{q}_i^\star - H^\star(q^\star, p^\star)]dt} \prod_{t} \prod_{i=1}^{N-J} \frac{dq_i^\star(t)dp_i^\star(t)}{2\pi} \qquad (2.16)$$

proving the assertion.

We mention the two main problems with the above arguments. First, adequate gauge conditions in some cases cannot be found. Although it might be possible to choose satisfactory gauge conditions locally, global gauge conditions are, more often than not, hard to obtain. This phenomenon is known as Gribov obstruction [7]. Second, the canonical transformations carried out in the proof of the assertion cannot be justified within the path integral formulation [17]. These are the two main reasons to study alternate schemes to handle constrained systems in a path integral framework. We now discuss the quantization of systems that have both first-class and second-class constraints.

2.2 Systems with Second-class Constraints

In this section we present Senjanovic's formalism of quantization of systems where second-class constraints are present [15]. This is a generalization of Faddeev's results described in the previous section. Consider a problem with the following constraints

$$\phi_a(q, p) = 0, \qquad \theta_b(q, p) = 0, \qquad a = 1, 2, ..., m, \qquad b = 1, 2, ..., 2n. \qquad (2.17)$$

The constraints ϕ_a's which are first-class, and the second-class θ_b's, determine the constraint manifold M. We assume the constraints are independent and irreducible in

the sense that an arbitrary function $f(q,p)$ which vanishes on the constraint surface can be expressed as

$$f(q,p) = c_a(q,p)\phi_a(q,p) + d_b(q,p)\theta_b(q,p). \tag{2.18}$$

Since the ϕ_a's are first-class and the θ_b's second-class we have the following relations amongst the constraints

$$\{\phi_a, \phi_b\} = c_{abc}\phi_c + d_{abe}\theta_e$$
$$\{\phi_a, \theta_f\} = g_{afm}\phi_m + h_{afk}\theta_k. \tag{2.19}$$

Also, the determinant of the Poisson brackets of the second-class constraints amongst themselves on the surface M is nonvanishing, i.e.,

$$\det ||\{\theta_a, \theta_b\}||_M \neq 0. \tag{2.20}$$

For the system under consideration, since there are m first-class constraints, we will have m gauge fixing conditions, one for each gauge generator, for reasons noted in the previous section. Thus, an observable $f(q,p)$ is uniquely determined by its value on a submanifold $\Gamma^\star$ of dimension $(2N - m - 2n) - m = 2(N - m - n)$. Hence, the submanifold $\Gamma^\star$ is specified by the constraints (17) together with the gauge conditions

$$\chi_a(q,p) = 0, \qquad a = 1, 2, ..., m. \tag{2.21}$$

The gauge conditions must be such that they satisfy the following two equations

$$\{\chi_a, \chi_b\} = 0, \qquad \det ||\{\chi_a, \phi_b\}|| \neq 0. \tag{2.22}$$

With the above gauge conditions the expression for the matrix element of the evolution operator is given by

$$\int e^{i\int_0^T [p_i\dot{q}_i - H(q,p)]dt} \prod_t d\mu(q(t)p(t)) \tag{2.23}$$

where the integration measure at each time slice in the expression above is given by

$$d\mu(t) = \det ||\{\chi_a, \phi_b\}|| [\det ||\{\theta_c, \theta_d\}||]^{\frac{1}{2}} \prod_{a=1}^{m} \delta(\chi_a)\delta(\phi_a) \prod_{c=1}^{2n} \delta(\theta_c) \prod_{i=1}^{N} dq_i(t)dp_i(t).$$

(2.24)

We will not discuss the proof of this assertion here, which the interested reader can find in Senjanovich [15]. It is noteworthy though that the proof, as in the case of first-class constraints, relies on the generally unjustifiable assumption that one can perform canonical transformations within a path integral.

2.3 Extended Coherent States

In this section we briefly discuss the use of extended coherent states (ECS) in the construction of path integrals developed by Klauder and Whiting [16]. Traditionally, coherent states have been constructed using a minimum number of quantum operators and classical variables appropriate for the problem under consideration. In constructing ECS one uses auxiliary quantum operators and associated classical variables. Such states, when used in the construction of path integrals, lead to extra path-space variables in the action. We will consider a simple model and using ECS show how the auxiliary variables could lead to the emergence of classical constraints. This section closely follows the contents of the paper by Klauder and Whiting mentioned above.

2.3.1 Basic Example of ECS

The conventional canonical coherent states based on an irreducible, self-adjoint representation of an Heisenberg pair of operators Q and P that satisfy $[Q, P] = i$, are given by

$$|p, q\rangle = e^{-iqP}e^{ipQ}|\eta\rangle = U(p, q)|\eta\rangle.$$

(2.25)

Here $|\eta\rangle$ is an arbitrary, normalized fiducial vector and $(p, q) \in \mathbb{R}^2$. For any $|\eta\rangle$ these states satisfy

$$1 = \int |p, q\rangle\langle p, q| \frac{dpdq}{2\pi} = \int U(p, q)|\eta\rangle\langle\eta|U^\dagger(p, q) \frac{dpdq}{2\pi} \qquad (2.26)$$

a resolution of unity in terms of equally weighted projection operators these states make. If $\mathcal{H} = \mathcal{H}(P, Q)$ is the Hamiltonian for some quantum mechanical system, then the propagator admits a formal path integral expression in the form

$$\langle p'', q''|e^{-i\mathcal{H}T}|p', q'\rangle = \int e^{i\int[i\langle p,q|\frac{d}{dt}|p,q\rangle - \langle p,q|\mathcal{H}(P,Q)|p,q\rangle]dt} \mathcal{D}p\mathcal{D}q \qquad (2.27)$$

where $(p(0), q(0)) = (p', q')$ and $(p(T), q(T)) = (p'', q'')$ are the boundary conditions. This prescription suggests the interpretation of

$$I = \int [i\langle p, q|\frac{d}{dt}|p, q\rangle - \langle p, q|\mathcal{H}(P, Q)|p, q\rangle]dt \qquad (2.28)$$

as a classical action for the system [18].

We now introduce an example of an extended coherent state. Let $D = (QP + PQ)/2$ denote the dilation generator that satisfies the commutation relations $[Q, D] = iQ$ and $[P, D] = -iP$. Together with the commutator $[Q, P] = i$, it follows that (P, Q, D) make a three parameter Lie algebra. We now define a unitary operator

$$V(r) = e^{i(\ln r)D}, \qquad r > 0, \qquad (2.29)$$

and let $|\xi\rangle = V(r)|\eta\rangle$. One notices the following

$$1 = \int U(p, q)|\xi\rangle\langle\xi|U^\dagger(p, q) \frac{dpdq}{2\pi} = \int |p, q, r\rangle\langle p, q, r| \frac{dpdq}{2\pi} \qquad (2.30)$$

is true for all r. The states $|p, q, r\rangle \equiv U(p, q)V(r)|\eta\rangle$ are an example of extended coherent states. Now if $\sigma(r)$ is an arbitrary measure on r such that $\int d\sigma(r) = 1$, then it is obvious that

$$1 = \int |p, q, r\rangle\langle p, q, r| \frac{dpdq\sigma(r)}{2\pi} = \int |p, q, r\rangle\langle p, q, r|d\mu(p, q, r). \qquad (2.31)$$

This non-uniqueness of the measure in the resolution of unity is a salient feature of extended coherent states and is exploited advantageously in path integrals.

We now study the propagator in the ECS representation. Proceeding in a manner similar to the case of canonical coherent states, one can write a formal path integral expression for the propagator in the ECS representation as

$$\langle p'', q'', r'' | e^{-i\mathcal{H}T} | p', q', r' \rangle = \int e^{i \int [i\langle p,q,r | \frac{d}{dt} | p,q,r \rangle - \langle p,q,r | \mathcal{H}(P,Q) | p,q,r \rangle] dt} \mathcal{D}\mu(p,q,r),$$

(2.32)

where the choice of the non-unique measure $\mu(p,q,r)$ can depend on time. From the path integral point of view the expression

$$I = \int [i\langle p,q,r | \frac{d}{dt} | p,q,r \rangle - \langle p,q,r | \mathcal{H} | p,q,r \rangle] dt$$

(2.33)

assumes the role of the classical action. In this expression for the action there is no trace of which variables are fundamental and which auxiliary; this division is embodied in the choice of the measure $\mu(p,q,r)$.

2.3.2 Classical Limit

We will now consider a few simple examples to illustrate how constraints can arise in going to the classical limit from the ECS path integral. Using the notation $\langle \star \rangle = \langle \eta | \star | \eta \rangle$, we define the following quantities

$$P(r) = V^\dagger(r) P V(r), \qquad Q(r) = V^\dagger(r) Q V(r), \qquad p^\star = p + \langle P(r) \rangle,$$

$$q^\star = q + \langle Q(r) \rangle, \qquad \Delta Q = Q - \langle Q \rangle, \qquad \Delta P = P - \langle P \rangle.$$

(2.34)

Expressed in terms of these variables the action in (33) can be written as

$$I = \int [p^\star \dot{q}^\star - \langle Q(r) \rangle \dot{p} - \langle D \rangle (\dot{r}/r) - H(p^\star, q^\star, r)] dt$$

(2.35)

where in the equation above $H(p^\star, q^\star, r) = \langle p,q,r | \mathcal{H}(P,Q) | p,q,r \rangle$. We will now study the equations of motions obtained from this action.

We notice that the term containing the time derivative of r in the action above is a total derivative and can be discarded as far as the equations of motion are concerned. Thus, for the Harmonic oscillator where $2\mathcal{H} = P^2 + Q^2$, we find $2H = p^{\star 2} + q^{\star 2} + r^2 \langle (\Delta P)^2 \rangle + (1/r^2) \langle (\Delta Q)^2 \rangle$. It is clear that the stationary variation of the action with respect to r leads to the constraint

$$\frac{\partial H}{\partial r} = r \langle (\Delta P)^2 \rangle - (1/r^3) \langle (\Delta Q)^2 \rangle = 0. \tag{2.36}$$

So, even though the quantum corrections $\langle (\Delta P)^2 \rangle$ and $\langle (\Delta Q)^2 \rangle$ are of order $\hbar$, the solution for r obtained from

$$r^4 = \frac{\langle (\Delta P)^2 \rangle}{\langle (\Delta Q)^2 \rangle} \tag{2.37}$$

is effectively independent of $\hbar$ and could be required to hold even in the limit $\hbar \to 0$. Next, consider the case of the quartic potential, $2\mathcal{H} = P^2 + Q^4$. The 'classical Hamiltonian' is given by

$$2H = p^{\star 2} + q^{\star 4} + r^2 \langle (\Delta P)^2 \rangle + \frac{6q^{\star 2}}{r^2} \langle (\Delta Q)^2 \rangle + \frac{4q^{\star}}{r^3} \langle (\Delta Q)^3 \rangle + \frac{1}{r^4} \langle (\Delta Q)^4 \rangle. \tag{2.38}$$

Again, extremisation of the action leads to the constraint $(\partial H / \partial r) = 0$, with the solution $r = r(q^{\star})$. The auxiliary variable r and the shifted canonical variable $q^{\star}$ are inextricably inter-coupled. Elimination of this constraint will not change the classical dynamics, but will clearly break contact with the quantum theory because there is no reason to suppose that any measure exists which can preserve the resolution of unity and also be compatible with the constraint at the same time.

In the discussion above we went 'against the grain' by starting with a quantum theory and then obtaining a classical limit from it. We saw from the example considered that the choice of the coherent state extension can strongly effect what becomes the classical theory. It is evident though that the relation between the resultant classical theory and

the initial quantum theory is far from transparent. Also, the elimination of the classical constraint variable can effect the transition back to the quantum theory.

In the next two chapters we discuss how *Bicoherent States* can be used to handle systems with constraints. We will consider models with first-class constraints and those with second-class constraints. For clarity of the formalism we will consider systems with a finite number of degrees of freedom as already noted.

CHAPTER 3
BICOHERENT STATES AND FIRST-CLASS CONSTRAINTS

In this chapter we describe a new technique for handling theories with first-class constraints in a path integral framework. Recall that first-class constraints are those whose Poisson brackets with all other constraints vanish on the constraint surface. We consider a toy model which has been studied before [19,20]. The model has a primary and a secondary constraint, both first-class. We quantize the model and following Dirac identify the physical subspace [10]. Then, starting with an orthonormal basis in this subspace we build a projection operator which we use in the construction of a coherent-state path-integral representation of the propagator for our model. The projection operator, which leads to the introduction of the bicoherent states, gives us the correct measure for the path integral and also leads to the desired classical limit.

This chapter is organized as follows: In section 3.1, we introduce the classical and quantum description of the model and identify its physical subspace. In section 3.2, which contains the bulk of the formalism developed here, we construct the projection operator and use it to evaluate the propagator for the case of a quadratic potential. We then obtain the classical limit from this propagator. In section 3.3, we study the propagator for a quartic potential. Section 3.4 discusses the measure obtained for the path integral representation of the propagator. In appendix A, we summarize the main features of path integrals constructed using bicoherent states.

3.1 Toy Model

We consider the dynamical system described by the Lagrangian

$$L(\mathbf{x}, \dot{\mathbf{x}}, y, \dot{y}) = \frac{1}{2}(\dot{\mathbf{x}} - yT\mathbf{x})^2 - V(\mathbf{x}) \tag{3.1}$$

where $\mathbf{x} = (x_1, x_2)$, a two-dimensional vector, and y are dynamical variables. Also, $T = i\tau_2$ is a 2×2 matrix where τ_2 is a Pauli matrix. As a first step toward quantization we go over to the Hamiltonian formalism. The canonically conjugate momenta to the coordinates are

$$
\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}} - yT\mathbf{x}
$$
$$
\pi = \frac{\partial L}{\partial \dot{y}} = 0 \tag{3.2}
$$

and the canonical Hamiltonian is

$$
H = \frac{1}{2}\mathbf{p^2} + V(\mathbf{x}) + y\mathbf{p}T\mathbf{x}. \tag{3.3}
$$

Therefore, we have a mechanical system with one primary constraint $\pi = 0$. We want our primary constraint to hold at all times, so, we require

$$
\dot{\pi} = \{\pi, H\} = -\mathbf{p}T\mathbf{x} = -\sigma = 0 \tag{3.4}
$$

i.e., we have a secondary constraint, $\sigma = 0$. We note that $\sigma = (p_1 x_2 - p_2 x_1)$ is just the generator of rotations in two dimensions. Both constraints in our problem are first-class since $\{\sigma, \pi\} = 0$. Also, there are no further constraints in the problem because $\dot{\sigma} = \{\sigma, H\} = 0$. Thus, our model which has two first-class constraints has only one physical degree of freedom which can be identified as follows: Performing the canonical transformation $(\mathbf{x}, \mathbf{p}) \rightarrow (r, \theta, p_r, p_\theta)$ where (r, θ) are polar coordinates and (p_r, p_θ) are momenta conjugate to them respectively, we find that (r, p_r) are gauge invariant and can be taken as the physical variables.

In the case where one is interested in the most general physically permissible motion one should allow for an arbitrary gauge transformation to be performed while the system is dynamically evolving in time. Hence, we add to our Hamiltonian the two first-class constraints multiplied by their corresponding Lagrange multipliers and obtain the extended

Hamiltonian

$$H = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{x}) + y\mathbf{p}T\mathbf{x} + u\mathbf{p}T\mathbf{x} + v\pi, \tag{3.5}$$

where u and v are Lagrange multipliers [12]. We use the extended Hamiltonian in the construction of the propagator and henceforth will refer to it as just the Hamiltonian unless otherwise specified.

The transition to the quantum description of the system is made by promoting the dynamical variables $\mathbf{x}, \mathbf{p}, y, \pi$ and the Lagrange multipliers u, v to operators. We will use the same symbol to represent the classical variables and their corresponding quantum operators since the quantity being referred to will be clear from the context. At present, we consider the harmonic oscillator potential $V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^2$, and so, our Hamiltonian is

$$H = \frac{1}{2}\mathbf{p}^2 + \frac{1}{2}\mathbf{x}^2 + y\mathbf{p}T\mathbf{x} + u\mathbf{p}T\mathbf{x} + v\pi. \tag{3.6}$$

It is useful to express our quantum mechanical problem in a second-quantized representation. Thus, using creation and annihilation operators we define

$$x_j = \frac{(a_j + a_j^\dagger)}{\sqrt{2}}, \qquad p_j = \frac{(a_j - a_j^\dagger)}{\sqrt{2}i}, \qquad\qquad j = 1, 2$$

$$y = \frac{(a_3 + a_3^\dagger)}{\sqrt{2}}, \qquad \pi = \frac{(a_3 - a_3^\dagger)}{\sqrt{2}i}, \qquad u = \frac{(a_4 + a_4^\dagger)}{\sqrt{2}}, \qquad v = \frac{(a_5 + a_5^\dagger)}{\sqrt{2}}, \tag{3.7}$$

and adopt the normal-ordered Hamiltonian $\mathcal{H} = \; : H : \;$ given by

$$\mathcal{H} = a_1^\dagger a_1 + a_2^\dagger a_2 + i\frac{(a_3^\dagger + a_3)}{\sqrt{2}}(a_1^\dagger a_2 - a_2^\dagger a_1) + i\frac{(a_4^\dagger + a_4)}{\sqrt{2}}(a_1^\dagger a_2 - a_2^\dagger a_1) \\ + \frac{(a_5^\dagger + a_5)}{\sqrt{2}}\frac{(a_3 - a_3^\dagger)}{\sqrt{2}i} \tag{3.8}$$

where $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$ and $i, j = 1, 2, 3, 4, 5$. An orthonormal basis for the Hilbert space under consideration is given by the oscillator occupation number states

$$\frac{(a_1^\dagger)^l}{\sqrt{l!}} \frac{(a_2^\dagger)^m}{\sqrt{m!}} \frac{(a_3^\dagger)^n}{\sqrt{n!}} \frac{(a_4^\dagger)^r}{\sqrt{r!}} \frac{(a_5^\dagger)^s}{\sqrt{s!}} |0\rangle. \tag{3.9}$$

Given the quantization prescription and the Hilbert space indicated above, we would now like to identify the physical subspace which respects the constraints as quantum operators. First, consider the subspace in which the operators $(\mathbf{x}, \mathbf{p})$ live, which in the occupation number representation corresponds to the space spanned by the vectors

$$\frac{(a_1^\dagger)^l}{\sqrt{l!}} \frac{(a_2^\dagger)^m}{\sqrt{m!}} |0\rangle = |l, m\rangle; \tag{3.10}$$

in this subspace, the physical states $|\phi\rangle$ are singled out by the condition

$$\sigma|\phi\rangle = \mathbf{p}T\mathbf{x}|\phi\rangle = i(a_1^\dagger a_2 - a_2^\dagger a_1)|\phi\rangle = 0. \tag{3.11}$$

Thus, the physical basis in this subspace is obtained by applying to the vacuum state $|0\rangle$ polynomials in $a_1^\dagger$ and $a_2^\dagger$ that commute with $\sigma = i(a_1^\dagger a_2 - a_2^\dagger a_1)$. The only such independent invariant polynomial is $(a_1^{\dagger 2} + a_2^{\dagger 2})$ i.e.,

$$[\sigma, (a_1^{\dagger 2} + a_2^{\dagger 2})] = 0. \tag{3.12}$$

Hence, an orthonormal basis in the physical subspace under consideration is given by

$$|\phi_k\rangle = \frac{(a_1^{\dagger 2} + a_2^{\dagger 2})^k}{2^k k!}|0\rangle, \qquad k = 0, 1, 2, \dots . \tag{3.13}$$

Notice that these states are also the energy eigenstates for the canonical Hamiltonian with a quadratic potential in (3) with eigenvalues $2, 4, 6, \dots$ in appropriate units [20].

Second, the physical states in the subspace where (y, π) operate is similarly determined by the condition $\pi|\phi\rangle = 0$. In the Fock space representation we replace this by the weaker requirement $\pi^{(-)}|\phi\rangle = 0$, where $\pi^{(-)}$ is the annihilation part of π. Thus, in this subspace the physical state is just the vacuum state $|0\rangle$. Hence, our physical states invariant under the two gauge transformations are

$$|\psi_k\rangle = \frac{(a_1^{\dagger 2} + a_2^{\dagger 2})^k}{2^k k!}|0, 0, 0\rangle. \tag{3.14}$$

These states form an orthonormal basis for the physical space which is a subspace of the Hilbert space spanned by the vectors in (9).

3.2 The Path Integral

The principal object represented by the path integral is the propagator. Before proceeding to the construction of the propagator for our toy model with two first-class constraints, we will briefly recall the construction of the propagator in the canonical coherent-state representation for systems with a single degree of freedom and without constraints. In the canonical coherent-state representation the propagator is given by

$$
\begin{aligned}
\langle z''|e^{-iT\mathcal{H}}|z'\rangle &= \int \langle z''|e^{-i\epsilon\mathcal{H}}|z_N\rangle\langle z_N|e^{-i\epsilon\mathcal{H}}|z_{N-1}\rangle...\langle z_1|e^{-i\epsilon\mathcal{H}}|z'\rangle \prod_{n=1}^{N} \frac{d^2 z_n}{\pi} \\
&= \int e^{i\int_0^T [\frac{1}{2}(p\dot{q}-q\dot{p})-H(p,q)]dt} \mathcal{D}p\mathcal{D}q,
\end{aligned}
\tag{3.15}
$$

where $H(p,q) = \langle p,q|\mathcal{H}|p,q\rangle$. The states $|z\rangle = |p,q\rangle$ are canonical coherent-states and are given by

$$
|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle,
\tag{3.16}
$$

and $z = (q+ip)/\sqrt{2}$. Also, the state $|n\rangle$ is the nth excited harmonic oscillator eigenstate. In (15) the resolution of unity

$$
1 = \sum_{n=0}^{\infty} |n\rangle\langle n| = \int |z\rangle\langle z|\frac{d^2 z}{\pi}
\tag{3.17}
$$

has been used at every time slice during the construction of the path integral [18].

The principal premise of this thesis is that in the construction of the path integral representation of the propagator for a constrained system, rather than the resolution of unity, one should use a projection operator which ensures that at every infinitesimal time step forward the evolving state is projected onto the physical subspace.

3.2.1 Projection Operator

We shall now construct the appropriate projection operator. In the space spanned by the basis vectors

$$|l, m, n\rangle = \frac{(a_1^\dagger)^l (a_2^\dagger)^m (a_3^\dagger)^n}{\sqrt{l!m!n!}} |0, 0, 0\rangle \tag{3.18}$$

the physical subspace is spanned by the orthonormal vectors

$$|\psi_k\rangle = \frac{(a_1^{\dagger 2} + a_2^{\dagger 2})^k}{2^k k!} |0, 0, 0\rangle, \tag{3.19}$$

as already noted. Hence, a projection operator which will project vectors onto the physical subspace is

$$P' = \sum_{k=0}^{\infty} |\psi_k\rangle\langle\psi_k|. \tag{3.20}$$

In order to use this projection operator in a path integral, we will write it as an integral in a fashion similar to how the unit operator is written as $1 = \int |z\rangle\langle z| d^2z/\pi$, and to this end we note that

$$a_1^l a_2^m |\gamma, \delta\rangle = \gamma^l \delta^m |\gamma, \delta\rangle,$$

$$\langle\alpha, \beta| a_1^{\dagger l} a_2^{\dagger m} = \langle\alpha, \beta| \alpha^{\star l} \beta^{\star m}, \tag{3.21}$$

where $|\alpha, \beta\rangle$ and $|\gamma, \delta\rangle$ are canonical coherent-states. We now show how to write our projection operator in an integral representation:

$$\begin{aligned}
P' &= \sum_{k=0}^{\infty} |\psi_k\rangle\langle\psi_k| \\
&= \sum_{k=0}^{\infty} \int \frac{d^2\alpha \, d^2\beta \, d^2\eta}{\pi^3} |\alpha, \beta, \eta\rangle\langle\alpha, \beta, \eta|\psi_k\rangle\langle\psi_k| \int \frac{d^2\gamma \, d^2\delta \, d^2\xi}{\pi^3} |\gamma, \delta, \xi\rangle\langle\gamma, \delta, \xi|,
\end{aligned} \tag{3.22}$$

in the above expression we have multiplied the projection operator by unity on either side and we obtain

$$\begin{aligned}
P' &= \sum_{k=0}^{\infty} \int \frac{d^2\alpha \, d^2\beta \, d^2\eta \, d^2\gamma \, d^2\delta \, d^2\xi}{\pi^6} |\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi|\langle\alpha, \beta, \eta|\psi_k\rangle\langle\psi_k|\gamma, \delta, \xi\rangle, \\
&= \sum_{k=0}^{\infty} \int \frac{d^2\alpha \, d^2\beta \, d^2\eta \, d^2\gamma \, d^2\delta \, d^2\xi}{\pi^6} |\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi| \frac{(\alpha^{\star 2} + \beta^{\star 2})^k}{2^k k!} \frac{(\gamma^2 + \delta^2)^k}{2^k k!} \\
&\quad \times e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2)},
\end{aligned} \tag{3.23}$$

where in (23) we have used the fact $\langle z|0\rangle = \langle 0|z\rangle = \exp(-\frac{1}{2}|z|^2)$. Next, using the Kronecker delta function in the form $\delta_{kl} = \int \exp\{i(k-l)\theta\}d\theta/2\pi$, we can write our projection operator as

$$P' = \sum_{k,l=0}^{\infty} \int \frac{d^2\alpha\, d^2\beta\, d^2\eta\, d^2\gamma\, d^2\delta\, d^2\xi}{\pi^6} \frac{d\theta}{2\pi} |\alpha,\beta,\eta\rangle\langle\gamma,\delta,\xi| \frac{(\alpha^{\star 2}+\beta^{\star 2})^k}{2^k k!} \frac{(\gamma^2+\delta^2)^l}{2^l l!}$$
$$\times e^{i(k-l)\theta} e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2+|\eta|^2+|\gamma|^2+|\delta|^2+|\xi|^2)} \quad (3.24)$$

and we notice that the summation above can be converted to an exponential. Finally, we get for our projection operator

$$P' = \int \frac{d^2\alpha\, d^2\beta\, d^2\eta\, d^2\gamma\, d^2\delta\, d^2\xi}{\pi^6} \frac{d\theta}{2\pi} |\alpha,\beta,\eta\rangle\langle\gamma,\delta,\xi|$$
$$\times e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2+|\eta|^2+|\gamma|^2+|\delta|^2+|\xi|^2)+\frac{(\alpha^{\star 2}+\beta^{\star 2})}{2}e^{i\theta}+\frac{(\gamma^2+\delta^2)}{2}e^{-i\theta}} . \quad (3.25)$$

The form of the projection operator in the above equation suggests the name *Bicoherent states*, where the term bicoherent alludes to the fact that the projection operator is represented by a weighted integral over independent coherent-state bras and kets.

Going back to the extended Hamiltonian in (6), recall that all primary and secondary constraints in the language of Dirac, appear in the Hamiltonian accompanied by their respective Lagrange multipliers and in the quantization process the Lagrange multipliers are also promoted to operators. Thus, to account for the Lagrange multipliers our projection operator becomes

$$P = \int \frac{d^2\alpha\, d^2\beta\, d^2\eta\, d^2\gamma\, d^2\delta\, d^2\xi\, d^2\rho\, d^2\sigma}{\pi^8} \frac{d\theta}{2\pi} |\alpha,\beta,\eta,\rho,\sigma\rangle\langle\gamma,\delta,\xi,\rho,\sigma|$$
$$\times e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2+|\eta|^2+|\gamma|^2+|\delta|^2+|\xi|^2)+\frac{(\alpha^{\star 2}+\beta^{\star 2})}{2}e^{i\theta}+\frac{(\gamma^2+\delta^2)}{2}e^{-i\theta}}, \quad (3.26)$$

where in the above expression $a_4|\alpha,\beta,\eta,\rho,\sigma\rangle = \rho|\alpha,\beta,\eta,\rho,\sigma\rangle$ and $a_5|\alpha,\beta,\eta,\rho,\sigma\rangle = \sigma|\alpha,\beta,\eta,\rho,\sigma\rangle$. One can easily verify that the operator P above satisfies the two defining properties of a projection operator, namely, $P^\dagger = P$ and $P^2 = P$.

We will use the projection operator in (26), in the construction of the path integral representation of the propagator in the next subsection. At this point, however, the interested reader may want to digress to the appendix A where we discuss the salient features of path integral representations constructed using bicoherent states.

3.2.2 Propagator

We are now equipped with the necessary tools to derive the path integral representation of the propagator. We will calculate the matrix element of the evolution operator between canonical coherent-states i.e., $\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle$. According to our premise, the propagator is

$$\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle$$
$$= \langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-i\epsilon\mathcal{H}} P_N e^{-i\epsilon\mathcal{H}} \ldots P_1 e^{-i\epsilon\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle, \tag{3.27}$$

where at each time slice we have inserted

$$P_n = \int \frac{d^2\alpha_n d^2\beta_n d^2\eta_n d^2\gamma_n d^2\delta_n d^2\xi_n d^2\rho_n d^2\sigma_n}{\pi^8} \frac{d\theta_n}{2\pi}$$
$$\times |\alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n\rangle\langle\gamma_n, \delta_n, \xi_n, \rho_n, \sigma_n| \tag{3.28}$$
$$\times e^{-\frac{1}{2}(|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\gamma_n|^2+|\delta_n|^2+|\xi_n|^2)+\frac{(\alpha_n^{\star 2}+\beta_n^{\star 2})}{2}e^{i\theta_n}+\frac{(\gamma_n^2+\delta_n^2)}{2}e^{-i\theta_n}}$$

the projection operator in (26). Also, $(N+1)\epsilon = T$ and $n = 1, 2, \ldots N$. Hence, the propagator now becomes

$$\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle =$$
$$\int \prod_{n=0}^{N} \langle\gamma_{n+1}, \delta_{n+1}, \xi_{n+1}, \rho_{n+1}, \sigma_{n+1}| e^{-i\epsilon\mathcal{H}} |\alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n\rangle \prod_{n=1}^{N} d\mu_n'', \tag{3.29}$$

where the measure at each time slice is

$$d\mu_n'' = \frac{d^2\alpha_n d^2\beta_n d^2\eta_n d^2\gamma_n d^2\delta_n d^2\xi_n d^2\rho_n d^2\sigma_n}{\pi^8} \frac{d\theta_n}{2\pi}$$
$$\times e^{-\frac{1}{2}(|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\gamma_n|^2+|\delta_n|^2+|\xi_n|^2)+\frac{(\alpha_n^{\star 2}+\beta_n^{\star 2})}{2}e^{i\theta_n}+\frac{(\gamma_n^2+\delta_n^2)}{2}e^{-i\theta_n}} \tag{3.30}$$

and the boundary conditions are

$$(\gamma_{N+1}, \delta_{N+1}, \xi_{N+1}, \rho_{N+1}, \sigma_{N+1}) = (\alpha'', \beta'', \eta'', \rho'', \sigma''),$$
$$(\alpha_0, \beta_0, \eta_0, \rho_0, \sigma_0) = (\alpha', \beta', \eta', \rho', \sigma'). \tag{3.31}$$

For small ϵ, we have, to order ϵ,

$$\langle\gamma_{n+1}, \delta_{n+1}, \xi_{n+1}, \rho_{n+1}, \sigma_{n+1}| e^{-i\epsilon\mathcal{H}} |\alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n\rangle$$
$$\simeq \langle\gamma_{n+1}, \delta_{n+1}, \xi_{n+1}, \rho_{n+1}, \sigma_{n+1}| [1 - i\epsilon\mathcal{H}] |\alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n\rangle \tag{3.32}$$
$$= \langle\gamma_{n+1}, \delta_{n+1}, \xi_{n+1}, \rho_{n+1}, \sigma_{n+1}|\alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n\rangle [1 - i\epsilon H_{n+1,n}]$$

where in the expression above

$$H_{n+1,n} = \frac{\langle \gamma_{n+1}, \delta_{n+1}, \xi_{n+1}, \rho_{n+1}, \sigma_{n+1} | \mathcal{H} | \alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n \rangle}{\langle \gamma_{n+1}, \delta_{n+1}, \xi_{n+1}, \rho_{n+1}, \sigma_{n+1} | \alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n \rangle}. \tag{3.33}$$

Thus, using the fact that $[1 - i\epsilon H_{n+1,n}] \simeq e^{-i\epsilon H_{n+1,n}}$ we are lead to the following expression, provided the integrals exist, for the propagator

$$\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle =$$

$$\int \prod_{n=0}^{N} \langle \gamma_{n+1}, \delta_{n+1}, \xi_{n+1}, \rho_{n+1}, \sigma_{n+1} | \alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n \rangle e^{-i\epsilon H_{n+1,n}} \prod_{n=1}^{N} d\mu_n''. \tag{3.34}$$

The canonical coherent-state overlap at each time slice in the above expression is

$$\langle \gamma_{n+1}, \delta_{n+1}, \xi_{n+1}, \rho_{n+1}, \sigma_{n+1} | \alpha_n, \beta_n, \eta_n, \rho_n, \sigma_n \rangle =$$

$$\times e^{\gamma_{n+1}^{\star}\alpha_n + \delta_{n+1}^{\star}\beta_n + \xi_{n+1}^{\star}\eta_n + \rho_{n+1}^{\star}\rho_n + \sigma_{n+1}^{\star}\sigma_n} \tag{3.35}$$

$$\times e^{-\frac{1}{2}(|\gamma_{n+1}|^2 + |\delta_{n+1}|^2 + |\xi_{n+1}|^2 + |\rho_{n+1}|^2 + |\sigma_{n+1}|^2 + |\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\rho_n|^2 + |\sigma_n|^2)}$$

and we notice that the factor $\exp\{(-1/2)(|\gamma_{n+1}|^2 + |\delta_{n+1}|^2 + |\xi_{n+1}|^2 + |\rho_{n+1}|^2 + |\sigma_{n+1}|^2 + |\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\rho_n|^2 + |\sigma_n|^2)\}$, except at the end points, can be absorbed in the measure $d\mu_n''$. Hence, our propagator becomes

$$\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle =$$

$$e^{-\frac{1}{2}(|\alpha''|^2 + |\beta''|^2 + |\eta''|^2 + |\rho''|^2 + |\sigma''|^2 + |\alpha'|^2 + |\beta'|^2 + |\eta'|^2 + |\rho'|^2 + |\sigma'|^2)} \times$$

$$\int e^{\sum_{n=0}^{N}[\gamma_{n+1}^{\star}\alpha_n + \delta_{n+1}^{\star}\beta_n + \xi_{n+1}^{\star}\eta_n + \rho_{n+1}^{\star}\rho_n + \sigma_{n+1}^{\star}\sigma_n - i\epsilon H_{n+1,n}]} \prod_{n=1}^{N} d\mu_n', \tag{3.36}$$

where the overall factor arose from the end points of the term absorbed in the measure, which at each time slice has changed slightly and is now given by

$$d\mu_n' = \frac{d^2\alpha_n d^2\beta_n d^2\eta_n d^2\gamma_n d^2\delta_n d^2\xi_n d^2\rho_n d^2\sigma_n}{\pi^8} \frac{d\theta_n}{2\pi}$$

$$\times e^{-(|\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\gamma_n|^2 + |\delta_n|^2 + |\xi_n|^2 + |\rho_n|^2 + |\sigma_n|^2) + \frac{(\alpha_n^{\star 2} + \beta_n^{\star 2})}{2}e^{i\theta_n} + \frac{(\gamma_n^2 + \delta_n^2)}{2}e^{-i\theta_n}}. \tag{3.37}$$

Our goal is to express the right hand side of (36) as a path integral and in preparation toward this objective we rewrite part of the exponent in the integrand of this equation

as follows:

$$\sum_{n=0}^{N}[\gamma_{n+1}^{\star}\alpha_n + \delta_{n+1}^{\star}\beta_n + \xi_{n+1}^{\star}\eta_n + \rho_{n+1}^{\star}\rho_n + \sigma_{n+1}^{\star}\sigma_n] = \sum_{n=0}^{N}\frac{1}{2}\{(\gamma_{n+1}^{\star} - \gamma_n^{\star})\alpha_n$$

$$- \gamma_{n+1}^{\star}(\alpha_{n+1} - \alpha_n) + (\delta_{n+1}^{\star} - \delta_n^{\star})\beta_n - \delta_{n+1}^{\star}(\beta_{n+1} - \beta_n) + (\xi_{n+1}^{\star} - \xi_n^{\star})\eta_n$$

$$- \xi_{n+1}^{\star}(\eta_{n+1} - \eta_n) + (\rho_{n+1}^{\star} - \rho_n^{\star})\rho_n - \rho_{n+1}^{\star}(\rho_{n+1} - \rho_n) + (\sigma_{n+1}^{\star} - \sigma_n^{\star})\sigma_n \quad (3.38)$$

$$- \sigma_{n+1}^{\star}(\sigma_{n+1} - \sigma_n)\} + \sum_{n=0}^{N}\frac{1}{2}\{\gamma_n^{\star}\alpha_n + \gamma_{n+1}^{\star}\alpha_{n+1} + \delta_n^{\star}\beta_n + \delta_{n+1}^{\star}\beta_{n+1} + \xi_n^{\star}\eta_n$$

$$+ \xi_{n+1}^{\star}\eta_{n+1} + \rho_n^{\star}\rho_n + \rho_{n+1}^{\star}\rho_{n+1} + \sigma_n^{\star}\sigma_n + \sigma_{n+1}^{\star}\sigma_{n+1}\}.$$

In the above equation, the terms $\gamma_0, \delta_0, \xi_0, \alpha_{N+1}, \beta_{N+1}$ and η_{N+1} have not yet been defined. The factors containing these terms cancel and so these terms can take on arbitrary values and are at our disposal. We shall assign them the following values: $(\gamma_0, \delta_0, \xi_0) = (\alpha', \beta', \eta')$ and $(\alpha_{N+1}, \beta_{N+1}, \eta_{N+1}) = (\alpha'', \beta'', \eta'')$. The choice of these special values will become clear presently. Going back to (38), we notice the second term on the right hand side of this equation can be absorbed in the measure, so, our propagator can be written as

$$\langle\alpha'', \beta'', \eta'', \rho'', \sigma''|e^{-iT\mathcal{H}}|\alpha', \beta', \eta', \rho', \sigma'\rangle = \int \exp\{\sum_{n=0}^{N}(\frac{1}{2}[(\gamma_{n+1}^{\star} - \gamma_n^{\star})\alpha_n$$

$$- \gamma_{n+1}^{\star}(\alpha_{n+1} - \alpha_n) + (\delta_{n+1}^{\star} - \delta_n^{\star})\beta_n - \delta_{n+1}^{\star}(\beta_{n+1} - \beta_n) + (\xi_{n+1}^{\star} - \xi_n^{\star})\eta_n$$

$$\quad (3.39)$$

$$- \xi_{n+1}^{\star}(\eta_{n+1} - \eta_n) + (\rho_{n+1}^{\star} - \rho_n^{\star})\rho_n - \rho_{n+1}^{\star}(\rho_{n+1} - \rho_n) + (\sigma_{n+1}^{\star} - \sigma_n^{\star})\sigma_n$$

$$- \sigma_{n+1}^{\star}(\sigma_{n+1} - \sigma_n)] - i\epsilon H_{n+1,n})\}\prod_{n=1}^{N} d\mu_n,$$

where the measure at each time slice gets modified again and now finally is

$$d\mu_n = \frac{d^2\alpha_n d^2\beta_n d^2\eta_n d^2\gamma_n d^2\delta_n d^2\xi_n d^2\rho_n d^2\sigma_n}{\pi^8}\frac{d\theta_n}{2\pi}\times$$

$$\quad (3.40)$$

$$e^{-(|\alpha_n|^2+|\beta_n|^2+|\eta_n|^2+|\gamma_n|^2+|\delta_n|^2+|\xi_n|^2)+\frac{(\alpha_n^{\star 2}+\beta_n^{\star 2})}{2}e^{i\theta_n}+\frac{(\gamma_n^2+\delta_n^2)}{2}e^{-i\theta_n}+\gamma_n^{\star}\alpha_n+\delta_n^{\star}\beta_n+\xi_n^{\star}\eta_n}.$$

Thus, interchanging the order of the limit and the integration in (39) we write for the propagator, in a formal way, the form it takes over continuous and differentiable paths as

$$\langle\alpha'', \beta'', \eta'', \rho'', \sigma''|e^{-iT\mathcal{H}}|\alpha', \beta', \eta', \rho', \sigma'\rangle =$$

$$\int e^{i\int_0^T\{\frac{i}{2}[\gamma^{\star}\dot{\alpha}-\dot{\gamma}^{\star}\alpha+\delta^{\star}\dot{\beta}-\dot{\delta}^{\star}\beta+\xi^{\star}\dot{\eta}-\dot{\xi}^{\star}\eta+\rho^{\star}\dot{\rho}-\dot{\rho}^{\star}\rho+\sigma^{\star}\dot{\sigma}-\dot{\sigma}^{\star}\sigma]-\langle\mathcal{H}\rangle\}dt} \quad (3.41)$$

$$\mathcal{D}\mu,$$

where $\mathcal{D}\mu = \prod_n d\mu_n$, and $\langle \mathcal{H} \rangle$ in the expression above is given by

$$\langle \mathcal{H} \rangle = \frac{\langle \gamma, \delta, \xi, \rho, \sigma | \mathcal{H} | \alpha, \beta, \eta, \rho, \sigma \rangle}{\langle \gamma, \delta, \xi, \rho, \sigma | \alpha, \beta, \eta, \rho, \sigma \rangle} = \gamma^\star \alpha + \delta^\star \beta$$
$$+ i\frac{(\xi^\star + \eta)}{\sqrt{2}}(\gamma^\star \beta - \delta^\star \alpha) + i\frac{(\rho^\star + \rho)}{\sqrt{2}}(\gamma^\star \beta - \delta^\star \alpha) + \frac{(\sigma^\star + \sigma)}{\sqrt{2}}\frac{(\eta - \xi^\star)}{\sqrt{2}i}, \quad (3.42)$$

where the Hamiltonian $\mathcal{H}$ is given in (8). Hence, according to (41) our phase-space action for continuous and differentiable paths is

$$S = \int_0^T \{(\frac{i}{2})[\gamma^\star \dot{\alpha} - \dot{\gamma}^\star \alpha + \delta^\star \dot{\beta} - \dot{\delta}^\star \beta + \xi^\star \dot{\eta} - \dot{\xi}^\star \eta + \rho^\star \dot{\rho} - \dot{\rho}^\star \rho + \sigma^\star \dot{\sigma} - \dot{\sigma}^\star \sigma]$$
$$- [\gamma^\star \alpha + \delta^\star \beta + i\frac{(\xi^\star + \eta)}{\sqrt{2}}(\gamma^\star \beta - \delta^\star \alpha) + i\frac{(\rho^\star + \rho)}{\sqrt{2}}(\gamma^\star \beta - \delta^\star \alpha) \quad (3.43)$$
$$+ \frac{(\sigma^\star + \sigma)}{\sqrt{2}}\frac{(\eta - \xi^\star)}{\sqrt{2}i}]\} dt$$

with the following boundary conditions

$$(\alpha(0), \beta(0), \eta(0), \rho(0), \sigma(0)) = (\gamma(0), \delta(0), \xi(0), \rho(0), \sigma(0))$$

$$= (\alpha', \beta', \eta', \rho', \sigma'),$$

$$(\alpha(T), \beta(T), \eta(T), \rho(T), \sigma(T)) = (\gamma(T), \delta(T), \xi(T), \rho(T), \sigma(T)) \quad (3.44)$$

$$= (\alpha'', \beta'', \eta'', \rho'', \sigma'').$$

3.2.3 Classical Limit

We will now study the classical equations of motion obtained from our phase-space action. However, before we do so, we add to our action a total time derivative which of course will not effect the equations of motion and write it as

$$S = \int_0^T \{i[\gamma^\star \dot{\alpha} + \delta^\star \dot{\beta} + \xi^\star \dot{\eta} + \rho^\star \dot{\rho} + \sigma^\star \dot{\sigma}] - [\gamma^\star \alpha + \delta^\star \beta$$
$$+ i\frac{(\xi^\star + \eta)}{\sqrt{2}}(\gamma^\star \beta - \delta^\star \alpha) + i\frac{(\rho^\star + \rho)}{\sqrt{2}}(\gamma^\star \beta - \delta^\star \alpha) + \frac{(\sigma^\star + \sigma)}{\sqrt{2}}\frac{(\eta - \xi^\star)}{\sqrt{2}i}]\} dt. \quad (3.45)$$

Notice our action is generally complex i.e., $S = S_1 + iS_2$. Varying S with respect to $\alpha, \beta, \eta, \gamma^\star, \delta^\star, \xi^\star, \rho^\star$, and $\sigma^\star$, while keeping their end points fixed, we get the following

equations of motion

$$\dot{\alpha} = \frac{\alpha}{i} + \frac{(\xi^\star + \eta)}{\sqrt{2}}\beta + \frac{(\rho^\star + \rho)}{\sqrt{2}}\beta, \qquad \dot{\gamma} = \frac{\gamma}{i} + \frac{(\xi + \eta^\star)}{\sqrt{2}}\delta + \frac{(\rho^\star + \rho)}{\sqrt{2}}\delta,$$

$$\dot{\beta} = \frac{\beta}{i} - \frac{(\xi^\star + \eta)}{\sqrt{2}}\alpha - \frac{(\rho^\star + \rho)}{\sqrt{2}}\alpha, \qquad \dot{\delta} = \frac{\delta}{i} - \frac{(\xi + \eta^\star)}{\sqrt{2}}\gamma - \frac{(\rho^\star + \rho)}{\sqrt{2}}\gamma,$$

$$\dot{\eta} = \frac{(\gamma^\star\beta - \delta^\star\alpha)}{\sqrt{2}} + \frac{(\sigma^\star + \sigma)}{2}, \qquad \dot{\xi} = \frac{(\alpha^\star\delta - \beta^\star\gamma)}{\sqrt{2}} + \frac{(\sigma^\star + \sigma)}{2},$$

$$\dot{\rho} = \frac{(\gamma^\star\beta - \delta^\star\alpha)}{\sqrt{2}}, \qquad \dot{\sigma} = \frac{(\xi^\star - \eta)}{2}.$$

$$(3.46)$$

Now, consider the paths $\alpha(t)$ and $\gamma(t)$; using the boundary conditions in (44) and the evolution equations above we find that

$$\alpha(0) = \gamma(0) = \alpha'$$

$$\dot{\alpha}(0) = \dot{\gamma}(0) = \frac{\alpha'}{i} + \frac{(\eta'^\star + \eta')}{\sqrt{2}}\beta' + \frac{(\rho'^\star + \rho')}{\sqrt{2}}\beta'. \qquad (3.47)$$

These are sufficient conditions for $\alpha(t) = \gamma(t)$ i.e., they evolve along identical paths. One can easily check that the pairs of paths $(\beta(t), \delta(t))$ and $(\eta(t), \xi(t))$ also start off with the same initial conditions and so we have $\beta(t) = \delta(t)$ and $\eta(t) = \xi(t)$. We will take up the equations for ρ and σ later; these equations determine the Lagrange multipliers.

To further our study of the classical equations let us define the complex quantities $\alpha, \beta, \eta, \gamma, \delta, \xi, \rho$, and σ as follows:

$$\alpha = \frac{(q_1 + ip_1)}{\sqrt{2}}, \qquad \beta = \frac{(q_2 + ip_2)}{\sqrt{2}}, \qquad \eta = \frac{(q_3 + ip_3)}{\sqrt{2}},$$

$$\gamma = \frac{(q_4 + ip_4)}{\sqrt{2}}, \qquad \delta = \frac{(q_5 + ip_5)}{\sqrt{2}}, \qquad \xi = \frac{(q_6 + ip_6)}{\sqrt{2}}, \qquad (3.48)$$

$$\rho = \frac{(q_7 + ip_7)}{\sqrt{2}}, \qquad \sigma = \frac{(q_8 + ip_8)}{\sqrt{2}}.$$

Now $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$ implies $(q_1(t), p_1(t), q_2(t), p_2(t), q_3(t), p_3(t)) = (q_4(t), p_4(t), q_5(t), p_5(t), q_6(t), p_6(t))$. Using this fact and the definitions in (48) we write the evolution equations (46) in terms of the variables $(q_1, p_1, q_2, p_2, q_3, p_3, q_7, p_7, q_8, p_8)$

as follows

$$\dot{q}_1 = p_1 + (q_3 + q_7)q_2, \qquad \dot{p}_1 = -q_1 + (q_3 + q_7)p_2,$$

$$\dot{q}_2 = p_2 - (q_3 + q_7)q_1, \qquad \dot{p}_2 = -q_2 - (q_3 + q_7)p_1,$$

$$\dot{q}_3 = q_8, \qquad \dot{p}_3 = (p_2 q_1 - p_1 q_2), \qquad \dot{q}_7 = 0,$$

(3.49)

$$\dot{p}_7 = (p_2 q_1 - p_1 q_2), \qquad \dot{q}_8 = 0, \qquad \dot{p}_8 = -p_3.$$

In order to compare the above equations of motion with those for the Hamiltonian $H = \frac{1}{2}(\mathbf{k}^2 + \mathbf{x}^2) + y\mathbf{k}T\mathbf{x} + \lambda\mathbf{k}T\mathbf{x} + \varsigma\pi$ we introduce a slightly extended classical phase-space action S', which will give us classical equations of motion in one-to-one correspondence with the equations in (49). The extended action is

$$S' = \int_0^T \{[k_1\dot{x}_1 + k_2\dot{x}_2 + \pi\dot{y} + p_\lambda\dot{\lambda} + p_\varsigma\dot{\varsigma}]$$

$$- [\frac{\mathbf{k}^2}{2} + \frac{\mathbf{x}^2}{2} + y\mathbf{k}T\mathbf{x} + \lambda\mathbf{k}T\mathbf{x} + \varsigma\pi]\}dt$$

(3.50)

where (λ, ς) are Lagrange multipliers and (p_λ, p_ς) are their respective conjugate momenta. The evolution equations obtained from the action S' are

$$\dot{x}_1 = k_1 + (y + \lambda)x_2, \qquad \dot{k}_1 = -x_1 + (y + \lambda)k_2,$$

$$\dot{x}_2 = k_2 - (y + \lambda)x_1, \qquad \dot{k}_2 = -x_2 - (y + \lambda)k_1,$$

$$\dot{y} = \varsigma, \quad \dot{\pi} = (x_1 k_2 - k_1 x_2), \qquad \dot{\lambda} = 0,$$

(3.51)

$$\dot{p}_\lambda = (x_1 k_2 - k_1 x_2), \qquad \dot{\varsigma} = 0, \quad \dot{p}_\varsigma = -\pi.$$

Comparing equations in (49) and (51) we see that if we make the identification

$$(q_1, p_1, q_2, p_2, q_3, p_3, q_7, p_7, q_8, p_8) \leftrightarrow (x_1, k_1, x_2, k_2, x_3, k_3, \lambda, p_\lambda, \varsigma, p_\varsigma) \qquad (3.52)$$

the two sets of equations would be the same. Thus, we conclude that the action obtained from our quantum propagator gives us the desired classical evolution equations and hence the right classical limit.

We will now note two interesting features about the classical limit of the formalism developed here. First, substituting the definitions (48) in our complex action $S = S_1 + iS_2$

in (45), we obtain for the real part of our action

$$
\begin{aligned}
S_1 = \frac{1}{2} \int_0^T \{ & [(p_4 \dot{q}_1 - q_4 \dot{p}_1) + (p_5 \dot{q}_2 - q_5 \dot{p}_2) + (p_6 \dot{q}_3 - q_6 \dot{p}_3) + (p_7 \dot{q}_7 - q_7 \dot{p}_7) \\
& + (p_8 \dot{q}_8 - q_8 \dot{p}_8)] - [(q_4 q_1 + p_4 p_1) + (q_5 q_2 + p_5 p_2)] - q_8(p_3 + p_6) \\
& + \frac{1}{2}(p_6 - p_3)[(q_5 q_1 + p_5 p_1) - (q_4 q_2 + p_4 p_2)] + \frac{1}{2}(q_6 + q_3)[(p_5 q_1 - q_5 p_1) \\
& - (p_4 q_2 - q_4 p_2)] + q_7[(p_5 q_1 - q_5 p_1) - (p_4 q_2 - q_4 p_2)] \} dt
\end{aligned}
$$

(3.53)

while the imaginary part of our complex action is given by

$$
\begin{aligned}
S_2 = \frac{1}{2} \int_0^T \{ & [q_4 \dot{q}_1 + p_4 \dot{p}_1 + q_5 \dot{q}_2 + p_5 \dot{p}_2 + q_6 \dot{q}_3 + p_6 \dot{p}_3 + q_7 \dot{q}_7 + p_7 \dot{p}_7 \\
& + q_8 \dot{q}_8 + p_8 \dot{p}_8] + [(p_4 q_1 - q_4 p_1) + (p_5 q_2 - q_5 p_2)] + q_8(q_3 - q_6) \\
& - \frac{1}{2}(q_6 + q_3)[(q_4 q_2 + p_4 p_2) - (q_5 q_1 + p_5 p_1)] - \frac{1}{2}(p_6 - p_3)[(p_5 q_1 - q_5 p_1) \\
& - (p_4 q_2 - q_4 p_2)] - q_7[(q_4 q_2 + p_4 p_2) - (q_5 q_1 + p_5 p_1)] \} dt.
\end{aligned}
$$

(3.54)

Extremising S_1 and S_2 in equations (53) and (54) respectively, while keeping the end points of the paths in them fixed we can obtain equations of motion for the dynamical variables $(q_1, p_1, q_2, p_2, q_3, p_3, q_4, p_4, q_5, p_5, q_6, p_6)$. We find that for each of these variables the evolution equations obtained from S_1 is *identical* to the one obtained from S_2. The second interesting fact is the following: we saw that on the classical trajectories $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$, which is equivalently stated as $(q_1(t), p_1(t), q_2(t), p_2(t), q_3(t), p_3(t)) = (q_4(t), p_4(t), q_5(t), p_5(t), q_6(t), p_6(t))$. Substituting this fact in (53) for S_1 we find that, up to a total derivative, S_1 on the classical trajectories reduces to

$$
\begin{aligned}
S_1 \to \int_0^T \{ & [p_1 \dot{q}_1 + p_2 \dot{q}_2 + p_3 \dot{q}_3 + p_7 \dot{q}_7 + p_8 \dot{q}_8] - \frac{1}{2}(q_1^2 + p_1^2) \\
& - \frac{1}{2}(q_2^2 + p_2^2) - q_3(p_1 q_2 - q_1 p_2) - q_7(p_1 q_2 - q_1 p_2) - q_8 p_3] \} dt
\end{aligned}
$$

(3.55)

exactly the standard classical phase-space action in (50). The imaginary part S_2 of the action with the above substitution becomes

$$S_2 \rightarrow \frac{1}{2} \int_0^T [q_1\dot{q}_1 + p_1\dot{p}_1 + q_2\dot{q}_2 + p_2\dot{p}_2 + q_3\dot{q}_3$$
$$+ p_3\dot{p}_3 + q_7\dot{q}_7 + p_7\dot{p}_7 + q_8\dot{q}_8 + p_8\dot{p}_8]dt, \qquad (3.56)$$

a complete surface term! So, S_2 on the classical trajectories gives rise to only an overall 'phase factor' in the propagator.

3.2.4 Constraint Hypersurface

Here we study the restrictions on the states over which the matrix element of the evolution operator is evaluated in the propagator. Consider the following equations from the set (49)

$$\dot{q}_3 = q_8, \qquad \dot{p}_3 = (p_2 q_1 - p_1 q_2), \qquad \dot{q}_7 = 0,$$
$$\dot{p}_7 = (p_2 q_1 - p_1 q_2), \qquad \dot{q}_8 = 0, \qquad \dot{p}_8 = -p_3. \qquad (3.57)$$

In the classical description of the model we are studying we had the two constraints $\pi = 0$ and $\mathbf{p}T\mathbf{x} = (p_1 x_2 - p_2 x_1) = 0$. So, in (57) we want $p_3 = \pi = 0$ and $\dot{p}_3 = \dot{p}_7 = 0$. Thus, the solutions to these equations are

$$q_8 = c_1, \qquad p_8 = c_2, \qquad q_7 = c_3,$$
$$p_7 = c_4, \qquad q_3 = c_1 t + c_5, \qquad p_3 = 0, \qquad (3.58)$$

where c_1, c_2, c_3, c_4, and c_5 are real constants which are determined by the particular classical solution one is interested in. Also, note that $\dot{p}_3(t) = (p_2 q_1 - p_1 q_2) = 0$ implies in particular $\dot{p}_3(0) = 2(\alpha'_R \beta'_I - \alpha'_I \beta'_R) = 0$ and $\dot{p}_3(T) = 2(\alpha''_R \beta''_I - \alpha''_I \beta''_R) = 0$, where (α_R, β_R) and (α_I, β_I) are the real and imaginary parts of (α, β) respectively. These restrictions on (α', β') and (α'', β'') can be stated alternatively as

$$(\alpha'^\star \beta' - \alpha' \beta'^\star) = 0, \qquad (\alpha''^\star \beta'' - \alpha'' \beta''^\star) = 0. \qquad (3.59)$$

Thus, in our propagator the states $|\alpha', \beta', \eta', \rho', \sigma'\rangle$ and $|\alpha'', \beta'', \eta'', \rho'', \sigma''\rangle$ are not arbitrary but must be restricted, as discussed above, to ensure that the system remains on the

constraint hypersurface in the classical limit. Hence, our propagator is

$$\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle =$$

$$\langle \alpha'', \beta'', \frac{c_1 T + c_5}{\sqrt{2}}, \frac{c_3 + ic_4}{\sqrt{2}}, \frac{c_1 + ic_2}{\sqrt{2}} | e^{-iT\mathcal{H}} | \alpha', \beta', \frac{c_5}{\sqrt{2}}, \frac{c_3 + ic_4}{\sqrt{2}}, \frac{c_1 + ic_2}{\sqrt{2}} \rangle, \quad (3.60)$$

where (α', β') and (α'', β'') are restricted as noted in (59).

3.3 The Quartic Potential

For completeness we shall consider the quartic potential $V(\mathbf{x}) = \frac{1}{4}(\mathbf{x}^2)^2$ and show

that we again obtain the correct classical limit by following the quantization procedure

outlined in this chapter. The Hamiltonian now is $H = \frac{1}{2}\mathbf{p}^2 + \frac{1}{4}(\mathbf{x}^2)^2 + y\mathbf{p}T\mathbf{x} + u\mathbf{p}T\mathbf{x} + v\pi$.

In the second-quantized notation the normal-ordered form of our Hamiltonian is

$$\mathcal{H} =: (-\frac{1}{4})[(a_1 - a_1^\dagger)^2 + (a_2 - a_2^\dagger)^2] + \frac{1}{16}[(a_1 + a_1^\dagger)^2 + (a_2 + a_2^\dagger)^2]^2$$

$$+ i\frac{(a_3 + a_3^\dagger)}{\sqrt{2}}(a_1^\dagger a_2 - a_2^\dagger a_1) + i\frac{(a_4 + a_4^\dagger)}{\sqrt{2}}(a_1^\dagger a_2 - a_2^\dagger a_1) + \frac{(a_5 + a_5^\dagger)}{\sqrt{2}}\frac{(a_3 - a_3^\dagger)}{\sqrt{2}i} : . $$

$$(3.61)$$

The propagator for canonical coherent-states is again

$$\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle =$$

$$\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-i\epsilon\mathcal{H}} P_N e^{-i\epsilon\mathcal{H}} P_{N-1} e^{-i\epsilon\mathcal{H}} ... P_1 e^{-i\epsilon\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle, \quad (3.62)$$

where P_n is the projection operator in (26). Interchanging the order of the limit and the

integration as usual we formally write the propagator as

$$\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle = \int e^{iS} \mathcal{D}\mu, \quad (3.63)$$

where the action, which is complex, is given for continuous and differentiable paths up

to a surface term by

$$S = \int_0^T \{ i[\gamma^\star \dot{\alpha} + \delta^\star \dot{\beta} + \xi^\star \dot{\eta} + \rho^\star \dot{\rho} + \sigma^\star \dot{\sigma}] + \frac{1}{4}[(\alpha - \gamma^\star)^2 + (\beta - \delta^\star)^2]$$

$$- \frac{1}{16}[(\alpha + \gamma^\star)^2 + (\beta + \delta^\star)^2]^2 - i[\frac{(\xi^\star + \eta)}{\sqrt{2}} + \frac{(\rho^\star + \rho)}{\sqrt{2}}](\gamma^\star \beta - \delta^\star \alpha) \quad (3.64)$$

$$- \frac{(\sigma^\star + \sigma)}{\sqrt{2}} \frac{(\eta - \xi^\star)}{\sqrt{2}i} \} dt$$

with boundary conditions specified in (44). Extremising S we obtain the following equations of motion; variation with respect to $\gamma^\star$ and α lead to

$$\dot{\alpha} = \frac{(\alpha - \gamma^\star)}{2i} + \frac{1}{4i}(\alpha + \gamma^\star)[(\alpha + \gamma^\star)^2 + (\beta + \delta^\star)^2] + \frac{\beta}{\sqrt{2}}[\xi^\star + \eta + \rho^\star + \rho],$$

$$\dot{\gamma} = \frac{(\gamma - \alpha^\star)}{2i} + \frac{1}{4i}(\gamma + \alpha^\star)[(\gamma + \alpha^\star)^2 + (\delta + \beta^\star)^2] + \frac{\delta}{\sqrt{2}}[\xi + \eta^\star + \rho^\star + \rho],$$

(3.65)

variation with respect to $\delta^\star$ and β leads to

$$\dot{\beta} = \frac{(\beta - \delta^\star)}{2i} + \frac{1}{4i}(\beta + \delta^\star)[(\alpha + \gamma^\star)^2 + (\beta + \delta^\star)^2] - \frac{\alpha}{\sqrt{2}}[\xi^\star + \eta + \rho^\star + \rho],$$

$$\dot{\delta} = \frac{(\delta - \beta^\star)}{2i} + \frac{1}{4i}(\delta + \beta^\star)[(\gamma + \alpha^\star)^2 + (\delta + \beta^\star)^2] - \frac{\gamma}{\sqrt{2}}[\xi + \eta^\star + \rho^\star + \rho],$$

(3.66)

and finally variation with respect to $\xi^\star$ and η lead to

$$\dot{\eta} = \frac{(\gamma^\star\beta - \delta^\star\alpha)}{\sqrt{2}} + \frac{(\sigma^\star + \sigma)}{2}, \qquad \dot{\xi} = \frac{(\alpha^\star\delta - \beta^\star\gamma)}{\sqrt{2}} + \frac{(\sigma^\star + \sigma)}{2}. \qquad (3.67)$$

Next, consider the trajectories $\alpha(t)$ and $\gamma(t)$; using (65) and the boundary conditions in (44) we find that for these paths

$$\alpha(0) = \gamma(0) = \alpha',$$

$$\dot{\alpha}(0) = \dot{\gamma}(0) = \{\frac{(\alpha' - \alpha'^\star)}{2i} + \frac{(\alpha' + \alpha'^\star)}{4i}[(\alpha' + \alpha'^\star)^2 + (\beta' + \beta'^\star)^2]$$

$$+ \frac{\beta'}{\sqrt{2}}[(\eta'^\star + \eta') + (\rho'^\star + \rho')]\}. \qquad (3.68)$$

They have identical initial conditions and hence evolve along the same paths, i.e. $\alpha(t) = \gamma(t)$. Similarly it can be confirmed easily that $\beta(t) = \delta(t)$ and $\eta(t) = \xi(t)$. So, using the fact that $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$ implies $(q_1(t), p_1(t), q_2(t), p_2(t), q_3(t), p_3(t)) = (q_4(t), p_4(t), q_5(t), p_5(t), q_6(t), p_6(t))$, and the definitions in (48) we can write the evolution equations (65–67) in terms of the variables $(q_1, p_1, q_2, p_2, q_3, p_3, q_7, p_7, q_8, p_8)$ and we get the following equations

$$\dot{q_1} = p_1 + q_2(q_3 + q_7), \qquad \dot{q_2} = p_2 - q_1(q_3 + q_7),$$

$$\dot{p_1} = -q_1(q_1^2 + q_2^2) + p_2(q_3 + q_7), \qquad \dot{p_3} = (q_1 p_2 - q_2 p_1),$$

$$\dot{p_2} = -q_2(q_1^2 + q_2^2) - p_1(q_3 + q_7), \qquad \dot{q_3} = 0, \quad \dot{q_7} = 0, \qquad (3.69)$$

$$\dot{p_7} = (q_1 p_2 - q_2 p_1), \qquad \dot{q_8} = 0, \quad \dot{p_8} = -p_3.$$

These are exactly the equations one would get from the classical phase-space action

$$S' = \int\limits_0^T \{[k_1\dot{x}_1 + k_2\dot{x}_2 + \pi\dot{y} + p_\lambda\dot{\lambda} + p_\varsigma\dot{\varsigma}]$$
$$- [\frac{k^2}{2} + \frac{1}{4}(\mathbf{x}^2)^2 + ykT\mathbf{x} + \lambda kT\mathbf{x} + \varsigma\pi]\}dt \tag{3.70}$$

if one makes the identification

$$(q_1, p_1, q_2, p_2, q_3, p_3, q_7, p_7, q_8, p_8) \leftrightarrow (x_1, k_1, x_2, k_2, x_3, k_3, \lambda, p_\lambda, \varsigma, p_\varsigma). \tag{3.71}$$

Hence, we see again that the action obtained from the quantum propagator gives us the desired classical equations of motion.

Now, let us substitute $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$ and the definitions (48) in (64). One can obtain the real and imaginary parts of the action evaluated on the classical trajectories. We get for the real part, up to a total derivative,

$$S_1 \rightarrow \int\limits_0^T \{[p_1\dot{q}_1 + p_2\dot{q}_2 + p_3\dot{q}_3 + p_7\dot{q}_7 + p_8\dot{q}_8] - \frac{1}{2}(p_1^2 + p_2^2)$$
$$- \frac{1}{4}(q_1^2 + q_2^2)^2 - (q_3 + q_7)(p_1q_2 - q_1p_2) - q_8p_3\}dt \tag{3.72}$$

exactly the classical phase-space action of equation (70). The imaginary part reduces to

$$S_2 \rightarrow \frac{1}{2}\int\limits_0^T [q_1\dot{q}_1 + p_1\dot{p}_1 + q_2\dot{q}_2 + p_2\dot{p}_2 + q_3\dot{q}_3$$
$$+ p_3\dot{p}_3 + q_7\dot{q}_7 + p_7\dot{p}_7 + q_8\dot{q}_8 + p_8\dot{p}_8]dt, \tag{3.73}$$

a surface term. Once again, we find that the complex action obtained from the quantum propagator gives us the correct classical evolution equations and when evaluated on the classical trajectories its real part is exactly equal to the classical phase-space action evaluated on the same paths and the imaginary part is just a surface term.

3.4 The Measure

We would now like to make the point that the procedure developed here for constructing the path integral for the propagator is merely a recipe for obtaining the correct

measure. We begin by noting that for a system with three dynamical degrees of freedom the unit operator is given by

$$1 = \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{\pi^3} |z_1, z_2, z_3\rangle\langle z_1, z_2, z_3|, \tag{3.74}$$

but the unit operator can also be written as

$$\begin{aligned} 1 &= \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{\pi^3} |z_1, z_2, z_3\rangle\langle z_1, z_2, z_3| \int \frac{d^2 z_4 d^2 z_5 d^2 z_6}{\pi^3} |z_4, z_5, z_6\rangle\langle z_4, z_5, z_6| \\ &= \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{\pi^3} \frac{d^2 z_4 d^2 z_5 d^2 z_6}{\pi^3} |z_1, z_2, z_3\rangle\langle z_4, z_5, z_6| \\ &\times e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 + |z_6|^2) + z_1^* z_4 + z_2^* z_5 + z_3^* z_6}, \end{aligned} \tag{3.75}$$

an integral over bicoherent states. So, for the Hamiltonian $\mathcal{H} = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{x}) + y\mathbf{p}T\mathbf{x} + u\mathbf{p}T\mathbf{x} + v\pi$ we are studying in this paper we could have written the unit operator as

$$\begin{aligned} 1 &= \int \frac{d^2\alpha d^2\beta d^2\eta d^2\gamma d^2\delta d^2\xi d^2\rho d^2\sigma}{\pi^8} |\alpha, \beta, \eta, \rho, \sigma\rangle\langle\gamma, \delta, \xi, \rho, \sigma| \\ &\times e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2) + \alpha^*\gamma + \beta^*\delta + \eta^*\xi}. \end{aligned} \tag{3.76}$$

Comparing this to the projection operator in (26) for our constrained system, which we reproduce below for convenience,

$$\begin{aligned} P &= \int \frac{d^2\alpha d^2\beta d^2\eta d^2\gamma d^2\delta d^2\xi d^2\rho d^2\sigma}{\pi^8} \frac{d\theta}{2\pi} |\alpha, \beta, \eta, \rho, \sigma\rangle\langle\gamma, \delta, \xi, \rho, \sigma| \\ &\times e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2) + \frac{(\alpha^{*2} + \beta^{*2})}{2} e^{i\theta} + \frac{(\gamma^2 + \delta^2)}{2} e^{-i\theta}} \end{aligned} \tag{3.77}$$

we see that the only difference between the projection operator and the unit operator is the measure over which the bicoherent states are integrated. Recall now the expression we obtained for the propagator using this projection operator at every time slice

$$\begin{aligned} \langle\alpha'', \beta'', \eta'', \rho'', \sigma''| e^{-iT\mathcal{H}} |\alpha', \beta', \eta', \rho', \sigma'\rangle &= \\ \int e^{i\int_0^T \{i[\gamma^*\dot{\alpha} + \delta^*\dot{\beta} + \xi^*\dot{\eta} + \rho^*\dot{\rho} + \sigma^*\dot{\sigma}] - \langle\mathcal{H}\rangle\}} \mathcal{D}\mu, \end{aligned} \tag{3.78}$$

where the discrete form of the measure is given by

$$\begin{aligned} \mathcal{D}\mu &= \prod_n \Big\{ \frac{d^2\alpha_n d^2\beta_n d^2\eta_n d^2\gamma_n d^2\delta_n d^2\xi_n d^2\rho_n d^2\sigma_n}{\pi^8} \frac{d\theta_n}{2\pi} \times \\ e^{-(|\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\gamma_n|^2 + |\delta_n|^2 + |\xi_n|^2) + \frac{(\alpha_n^{*2} + \beta_n^{*2})}{2} e^{i\theta_n} + \frac{(\gamma_n^2 + \delta_n^2)}{2} e^{-i\theta_n} + \gamma_n^*\alpha_n + \delta_n^*\beta_n + \xi_n^*\eta_n} \Big\}. \end{aligned} \tag{3.79}$$

On the other hand had we used the resolution of unity as written in (76) instead, the expression for our propagator would have been

$$
\langle \alpha'', \beta'', \eta'', \rho'', \sigma'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta', \rho', \sigma' \rangle =
$$
$$
\int e^{i \int_0^T \{ i[\gamma^\star \dot{\alpha} + \delta^\star \dot{\beta} + \xi^\star \dot{\eta} + \rho^\star \dot{\rho} + \sigma^\star \dot{\sigma}] - \langle \mathcal{H} \rangle \}} \mathcal{D}\mu_{unit}, \tag{3.80}
$$

and the measure would be

$$
\mathcal{D}\mu_{unit} = \prod_n \{ \frac{d^2\alpha_n d^2\beta_n d^2\eta_n d^2\gamma_n d^2\delta_n d^2\xi_n d^2\rho_n d^2\sigma_n}{\pi^8} \times
$$
$$
e^{-(|\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\gamma_n|^2 + |\delta_n|^2 + |\xi_n|^2) + \alpha_n^\star \gamma_n + \beta_n^\star \delta_n + \eta_n^\star \xi_n + \gamma_n^\star \alpha_n + \delta_n^\star \beta_n + \xi_n^\star \eta_n} \}. \tag{3.81}
$$

So, we see that both procedures would have lead to the same action for continuous and differentiable path, but with quite different measures! Since the actions are the same they would yield identical classical equations of motion, but the different measures would give different spectrums to the quantization, only one of them being correct, of course.

It must be noted that a general operator admits a bicoherent state representation according to

$$
O = \int |\alpha\rangle \langle \alpha | O | \beta \rangle \langle \beta | \frac{d^2\alpha d^2\beta}{\pi^2}. \tag{3.82}
$$

Such operators have been dealt with, for example, by R. J. Glauber [21]. What is novel in the present thesis is the use of such representations in path integral constructions for which in all previous coherent-state applications only weighted coherent-state projection operators have been used. The term 'Bicoherent States' was coined by Prof. J. R. Klauder.

CHAPTER 4
BICOHERENT STATES AND SECOND-CLASS CONSTRAINTS

In this chapter we extend the formalism of the previous chapter developed for systems with first-class constraints, to systems with second-class constraints. *The principal premise of this formalism is that in the construction of the path integral representation of the propagator for a constrained system one should use, rather than the resolution unity, a projection operator at every time slice. The projection operator is such that it ensures the evolving state, at every infinitesimal time step forward of its evolution, is projected onto the physical subspace.*

The present chapter discusses two distinct models. In section 4.1 we state the Hamiltonian for the first model that is being investigated and impose a constraint by hand. We construct the path integral representation of the propagator using the proposition stated above and using the phase-space action obtained from this propagator extract the classical equations of motion, which we show are the desired classical evolution equations. For the second model, which is discussed in section 4.2, we show that it has one primary and three secondary constraints, all second-class. We show our formalism again gives us the correct classical limit.

4.1 Toy Model 1

In this section we study the two dimensional Harmonic oscillator, described by the Hamiltonian $H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2)$, subject to constraints. Let us impose the constraint $\sigma_1 = (q_1 - q_2) = 0$ on our model. We want the constraint to hold at all times, so, we require

$$\dot{\sigma}_1 = \{\sigma_1, H\} = (p_1 - p_2) = \sigma_2 = 0, \tag{4.1}$$

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i.e. our model has a secondary constraint $\sigma_2 = 0$. We find that $\dot{\sigma}_2 = \{\sigma_2, H\} = -\sigma_1$, i.e. there are no further constraints in our model. Also, $\{\sigma_1, \sigma_2\} = 2$, hence, the two constraints are second-class.

In the quantization of systems with second-class constraints one finds that the constraints can be imposed as operator equations once the Poisson brackets are replaced by Dirac brackets. Thus, for the model being studied here, since the canonical Hamiltonian can be written as $H = \frac{1}{2}\left[(p_1 - p_2)^2 + 2p_1 p_2\right] + \frac{1}{2}\left[(q_1 - q_2)^2 + 2q_1 q_2\right]$, the physical Hamiltonian $H_{ph} = (H)_{\sigma_i = 0}$ may be written, preserving $1 \leftrightarrow 2$ symmetry, as

$$H_{ph} = p_1 p_2 + q_1 q_2. \tag{4.2}$$

We will use H_{ph} to identify the physical subspace of our model. For ease of writing we will use the same symbols to represent the classical variables and their corresponding quantum operators, since the quantity being referred to shall be clear from the context. Thus, using creation and annihilation operators we write

$$
\begin{aligned}
q_j &= \frac{(a_j + a_j^\dagger)}{\sqrt{2}}, & p_j &= \frac{(a_j - a_j^\dagger)}{\sqrt{2}i}, \\
[a_i, a_j^\dagger] &= \delta_{ij}, & [a_i, a_j] &= 0, & i, j = 1, 2,
\end{aligned}
\tag{4.3}
$$

we get : $H_{ph} := a_1^\dagger a_2 + a_2^\dagger a_1$. The eigenstates of this Hamiltonian are easily found by noting the fact that $[(a_1^\dagger a_2 + a_2^\dagger a_1), (a_1^\dagger + a_2^\dagger)] = (a_1^\dagger + a_2^\dagger)$ and thus, the normalized eigenstates are given by

$$|\phi_k\rangle = \frac{(a_1^\dagger + a_2^\dagger)^k}{(\sqrt{2})^k \sqrt{k!}}|0, 0\rangle. \tag{4.4}$$

These eigenstates span the physical subspace. Therefore, a projection operator that can be used in the construction of the path integral representation of the propagator is

$$P = \sum_{k=0}^{\infty} |\phi_k\rangle\langle\phi_k|. \tag{4.5}$$

This projection operator will project the evolving states onto the physical subspace. But to be able to use this projection operator in the construction of the path integral representation of the propagator we will first need to write it as an integral and we proceed as follows:

$$
\begin{aligned}
P &= \sum_{k=0}^{\infty} \int \frac{d^2\alpha d^2\beta}{\pi^2} |\alpha, \beta\rangle\langle\alpha, \beta||\phi_k\rangle\langle\phi_k| \int \frac{d^2\gamma d^2\delta}{\pi^2} |\gamma, \delta\rangle\langle\gamma, \delta|, \\
&= \sum_{k=0}^{\infty} \int \frac{d^2\alpha d^2\beta d^2\gamma d^2\delta}{\pi^4} |\alpha, \beta\rangle\langle\gamma, \delta|\langle\alpha, \beta|\phi_k\rangle\langle\phi_k|\gamma, \delta\rangle,
\end{aligned}
\tag{4.6}
$$

where we have multiplied the projection operator by unity on either side in the first line of the above equation. The states $|\alpha, \beta\rangle$ and $|\gamma, \delta\rangle$ are canonical coherent states where $a_1|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle$, $a_2|\alpha, \beta\rangle = \beta|\alpha, \beta\rangle$, and $a_1|\gamma, \delta\rangle = \gamma|\gamma, \delta\rangle$, $a_2|\gamma, \delta\rangle = \delta|\gamma, \delta\rangle$. Next, using the fact that $\langle 0, 0|\alpha, \beta\rangle = \langle\alpha, \beta|0, 0\rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)}$ we can write the projection operator as

$$
\begin{aligned}
P = \sum_{k=0}^{\infty} \int \frac{d^2\alpha d^2\beta d^2\gamma d^2\delta}{\pi^4} |\alpha, \beta\rangle\langle\gamma, \delta| \frac{(\alpha^\star + \beta^\star)^k}{(\sqrt{2})^k \sqrt{k!}} \frac{(\gamma + \delta)^k}{(\sqrt{2})^k \sqrt{k!}} \\
\times e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2)}.
\end{aligned}
\tag{4.7}
$$

In the above expression we see that the summation can be performed and so our projection operator can finally be written as

$$
P = \int \frac{d^2\alpha d^2\beta d^2\gamma d^2\delta}{\pi^4} |\alpha, \beta\rangle\langle\gamma, \delta| e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2) + \frac{(\alpha^\star + \beta^\star)}{\sqrt{2}} \frac{(\gamma + \delta)}{\sqrt{2}}}
\tag{4.8}
$$

a weighted integral over independent bras and kets, which are called bicoherent states [18].

4.1.1 Path Integral

We are now ready to construct the path integral representation for our canonical coherent-state propagator. As already noted, we should insert the projection operator constructed above at every time slice in the propagator rather than the resolution of unity, and so our propagator is given by

$$
\langle\alpha'', \beta''|e^{-iT\mathcal{H}}|\alpha', \beta'\rangle = \langle\alpha'', \beta''|e^{-i\epsilon\mathcal{H}}P_N e^{-i\epsilon\mathcal{H}}...P_1 e^{-i\epsilon\mathcal{H}}|\alpha', \beta'\rangle,
\tag{4.9}
$$

where $\mathcal{H} = \, : H : \, = a_1^\dagger a_1 + a_2^\dagger a_2$. Also, $(N+1)\epsilon = T$, and P_n, the projection operator at the nth time slice, is

$$P_n = \int \frac{d^2\alpha_n d^2\beta_n d^2\gamma_n d^2\delta_n}{\pi^4} |\alpha_n, \beta_n\rangle\langle\gamma_n, \delta_n|$$
$$\times\, e^{-\frac{1}{2}(|\alpha_n|^2+|\beta_n|^2+|\gamma_n|^2+|\delta_n|^2)+\frac{(\alpha_n^\star+\beta_n^\star)}{\sqrt{2}}\frac{(\gamma_n+\delta_n)}{\sqrt{2}}}, \tag{4.10}$$

with $n = 1, 2, ... N$. Thus, our propagator becomes

$$\langle\alpha'', \beta''|e^{-iT\mathcal{H}}|\alpha', \beta'\rangle = \int \prod_{n=0}^{N}\langle\gamma_{n+1}, \delta_{n+1}|e^{-i\epsilon\mathcal{H}}|\alpha_n, \beta_n\rangle \prod_{n=1}^{N} d\mu_n'', \tag{4.11}$$

where the measure at each time slice is

$$d\mu_n'' = \frac{d^2\alpha_n d^2\beta_n d^2\gamma_n d^2\delta_n}{\pi^4} e^{-\frac{1}{2}(|\alpha_n|^2+|\beta_n|^2+|\gamma_n|^2+|\delta_n|^2)+\frac{(\alpha_n^\star+\beta_n^\star)}{\sqrt{2}}\frac{(\gamma_n+\delta_n)}{\sqrt{2}}}, \tag{4.12}$$

and the boundary conditions are $(\gamma_{N+1}, \delta_{N+1}) = (\alpha'', \beta'')$ and $(\alpha_0, \beta_0) = (\alpha', \beta')$. In equation (11), for small ϵ, we have, correct to order ϵ,

$$\langle\gamma_{n+1}, \delta_{n+1}|e^{-i\epsilon\mathcal{H}}|\alpha_n, \beta_n\rangle \simeq \langle\gamma_{n+1}, \delta_{n+1}|[1 - i\epsilon\mathcal{H}]|\alpha_n, \beta_n\rangle$$
$$= \langle\gamma_{n+1}, \delta_{n+1}|\alpha_n, \beta_n\rangle[1 - i\epsilon H_{n+1,n}] \simeq \langle\gamma_{n+1}, \delta_{n+1}|\alpha_n, \beta_n\rangle e^{-i\epsilon H_{n+1,n}} \tag{4.13}$$

where in the expression above

$$H_{n+1,n} = \frac{\langle\gamma_{n+1}, \delta_{n+1}|\mathcal{H}|\alpha_n, \beta_n\rangle}{\langle\gamma_{n+1}, \delta_{n+1}|\alpha_n, \beta_n\rangle}. \tag{4.14}$$

Therefore, using the fact that $\langle\gamma_{n+1}, \delta_{n+1}|\alpha_n, \beta_n\rangle = \exp\{(-1/2)(|\gamma_{n+1}|^2 + |\delta_{n+1}|^2 + |\alpha_n|^2 + |\beta_n|^2) + \gamma_{n+1}^*\alpha_n + \delta_{n+1}^*\beta_n\}$ our propagator, provided the integrals exist, becomes

$$\langle\alpha'', \beta''|e^{-iT\mathcal{H}}|\alpha', \beta'\rangle$$
$$= \int \prod_{n=0}^{N} e^{-\frac{1}{2}(|\gamma_{n+1}|^2+|\delta_{n+1}|^2+|\alpha_n|^2+|\beta_n|^2)+\gamma_{n+1}^*\alpha_n+\delta_{n+1}^*\beta_n-i\epsilon H_{n+1,n}} \prod_{n=1}^{N} d\mu_n''. \tag{4.15}$$

Notice that the term $\exp\{(-1/2)(|\gamma_{n+1}|^2 + |\delta_{n+1}|^2 + |\alpha_n|^2 + |\beta_n|^2)\}$, except at the end points, can be absorbed in the measure and so our propagator now is given by

$$\langle\alpha'', \beta''|e^{-iT\mathcal{H}}|\alpha', \beta'\rangle$$
$$= e^{-\frac{1}{2}(|\alpha''|^2+|\beta''|^2+|\alpha'|^2+|\beta'|^2)} \int e^{\sum_{n=0}^{N}[\gamma_{n+1}^*\alpha_n+\delta_{n+1}^*\beta_n-i\epsilon H_{n+1,n}]} \prod_{n=1}^{N} d\mu_n', \tag{4.16}$$

where the overall factor arose from the end points of the term absorbed in the measure, which at each time slice has changed slightly and is now given by

$$d\mu'_n = \frac{d^2\alpha_n d^2\beta_n d^2\gamma_n d^2\delta_n}{\pi^4} e^{-(|\alpha_n|^2+|\beta_n|^2+|\gamma_n|^2+|\delta_n|^2)+\frac{(\alpha_n^\star+\beta_n^\star)}{\sqrt{2}}\frac{(\gamma_n+\delta_n)}{\sqrt{2}}}. \qquad (4.17)$$

Going back to (16), we note that the term $\sum_{n=0}^{N}[\gamma_{n+1}^\star\alpha_n + \delta_{n+1}^\star\beta_n]$ can be rewritten as follows,

$$\sum_{n=0}^{N}[\gamma_{n+1}^\star\alpha_n + \delta_{n+1}^\star\beta_n] = \sum_{n=0}^{N}\frac{1}{2}\{(\gamma_{n+1}^\star - \gamma_n^\star)\alpha_n - \gamma_{n+1}^\star(\alpha_{n+1} - \alpha_n)$$

$$+ (\delta_{n+1}^\star - \delta_n^\star)\beta_n - \delta_{n+1}^\star(\beta_{n+1} - \beta_n)\} + \sum_{n=0}^{N}\frac{1}{2}\{\gamma_n^\star\alpha_n + \gamma_{n+1}^\star\alpha_{n+1} \qquad (4.18)$$

$$+ \delta_n^\star\beta_n + \delta_{n+1}^\star\beta_{n+1}\}.$$

In the above expression the terms (γ_0, δ_0) and $(\alpha_{N+1}, \beta_{N+1})$ have not been defined yet. The factors containing these terms cancel and so these terms can take on arbitrary values. We assign them the following values: $(\gamma_0, \delta_0) = (\alpha', \beta')$ and $(\alpha_{N+1}, \beta_{N+1}) = (\alpha'', \beta'')$. The reason for the choice of these special values will become clear shortly. Notice that the second sum above can be absorbed in the measure, and so our propagator becomes,

$$\langle\alpha'', \beta''|e^{-iT\mathcal{H}}|\alpha', \beta'\rangle = \int \exp\{\sum_{n=0}^{N}(\frac{1}{2}[(\gamma_{n+1}^\star - \gamma_n^\star)\alpha_n - \gamma_{n+1}^\star(\alpha_{n+1} - \alpha_n)$$

$$+ (\delta_{n+1}^\star - \delta_n^\star)\beta_n - \delta_{n+1}^\star(\beta_{n+1} - \beta_n)] - i\epsilon H_{n+1,n})\}\prod_{n=1}^{N} d\mu_n, \qquad (4.19)$$

where the measure at each time slice has changed again and has now become

$$d\mu_n = \frac{d^2\alpha_n d^2\beta_n d^2\gamma_n d^2\delta_n}{\pi^4} e^{-(|\alpha_n|^2+|\beta_n|^2+|\gamma_n|^2+|\delta_n|^2)+\frac{(\alpha_n^\star+\beta_n^\star)}{\sqrt{2}}\frac{(\gamma_n+\delta_n)}{\sqrt{2}}+\gamma_n^\star\alpha_n+\delta_n^\star\beta_n}. \qquad (4.20)$$

Interchanging the order of the limit and integration in (19), we can formally write the propagator as an integral over continuous and differentiable paths as

$$\langle\alpha'', \beta''|e^{-iT\mathcal{H}}|\alpha', \beta'\rangle = \int e^{i\int_0^T\{\frac{1}{2}[\gamma^\star\dot{\alpha}-\dot{\gamma}^\star\alpha+\delta^\star\dot{\beta}-\dot{\delta}^\star\beta]-\langle\mathcal{H}\rangle\}dt}\mathcal{D}\mu, \qquad (4.21)$$

where in the above expression $\mathcal{D}\mu = \prod\limits_{n} d\mu_n$, and $\langle\mathcal{H}\rangle$ for the model under investigation is given by

$$\langle\mathcal{H}\rangle = \frac{\langle\gamma,\delta|\mathcal{H}|\alpha,\beta\rangle}{\langle\gamma,\delta|\alpha,\beta\rangle} = \gamma^{\star}\alpha + \delta^{\star}\beta. \tag{4.22}$$

Thus, the phase-space action according to (21) is

$$S = \int\limits_{0}^{T} \{(\frac{i}{2})[\gamma^{\star}\dot{\alpha} - \dot{\gamma}^{\star}\alpha + \delta^{\star}\dot{\beta} - \dot{\delta}^{\star}\beta] - [\gamma^{\star}\alpha + \delta^{\star}\beta]\}dt, \tag{4.23}$$

with the following boundary conditions

$$(\alpha(0),\beta(0)) = (\gamma(0),\delta(0)) = (\alpha',\beta')$$

$$(\alpha(T),\beta(T)) = (\gamma(T),\delta(T)) = (\alpha'',\beta''). \tag{4.24}$$

4.1.2 Classical Limit

We will now study the classical equations of motion obtained from our phase-space action in (23). However, before we do so, we add to our action a total time derivative which of course will not effect the equations of motion and write it as

$$S = \int\limits_{0}^{T} \{i[\gamma^{\star}\dot{\alpha} + \delta^{\star}\dot{\beta}] - [\gamma^{\star}\alpha + \delta^{\star}\beta]\}dt. \tag{4.25}$$

Our action, it is to be noted, is complex, i.e., $S = S_1 + iS_2$. Variation of S with respect to $\alpha, \beta, \gamma^{\star}$ and $\delta^{\star}$ while keeping the end points of these paths fixed gives us the following equations of motion

$$\dot{\alpha} = -i\alpha, \qquad \dot{\beta} = -i\beta, \qquad \dot{\gamma} = -i\gamma, \qquad \dot{\delta} = -i\delta. \tag{4.26}$$

Next, consider the paths $\alpha(t)$ and $\gamma(t)$; using the boundary conditions (24) and the evolution equations above, we have

$$\alpha(0) = \gamma(0) = \alpha'$$

$$\dot{\alpha}(0) = \dot{\gamma}(0) = -i\alpha'. \tag{4.27}$$

The above equations imply $\alpha(t) = \gamma(t)$. Similarly, one finds that $\beta(t)$ and $\delta(t)$ evolve along identical paths, i.e. $\beta(t) = \delta(t)$.

To further our study of the classical equations of motion we define the complex quantities α, β, γ, and δ as follows:

$$\alpha = \frac{(q_1 + ip_1)}{\sqrt{2}}, \qquad \beta = \frac{(q_2 + ip_2)}{\sqrt{2}}, \qquad \gamma = \frac{(q_3 + ip_3)}{\sqrt{2}}, \qquad \delta = \frac{(q_4 + ip_4)}{\sqrt{2}}. \quad (4.28)$$

Notice, the equality $(\alpha(t), \beta(t)) = (\gamma(t), \delta(t))$ implies $(q_1(t), p_1(t), q_2(t), p_2(t)) = (q_3(t), p_3(t), q_4(t), p_4(t))$. Using this fact and the definitions (28) we can write the evolution equations (26) in terms of the variables (q_1, p_1, q_2, p_2) as follows

$$\dot{q}_1 = p_1, \qquad \dot{p}_1 = -q_1, \qquad \dot{q}_2 = p_2, \qquad \dot{p}_2 = -q_2. \quad (4.29)$$

For the model being studied in this section the standard classical phase-space action is given by

$$S' = \int_0^T \{[k_1\dot{x}_1 + k_2\dot{x}_2] - [\frac{1}{2}(k_1^2 + k_2^2) + \frac{1}{2}(x_1^2 + x_2^2)]\}dt. \quad (4.30)$$

The equations of motion obtained from S' would be identical to the ones in (29) if we made the identification

$$(q_1, p_1, q_2, p_2) \leftrightarrow (x_1, k_1, x_2, k_2). \quad (4.31)$$

Thus, we see that the action obtained from our quantum propagator has given us the desired classical evolution equations and hence the correct classical limit.

It is interesting to note that just as in the case of the first class constraints, as discussed in chapter 3, if we substitute the definitions (28) in (23) and obtain the evolution equations for the variables $(q_1, p_1, q_2, p_2, q_3, p_3, q_4, p_4)$ from S_1 and S_2 we find that for each of these variables, the equation obtained from S_1 is exactly the same as the one obtained from S_2. Furthermore, on the classical trajectories where

$(q_1(t), p_1(t), q_2(t), p_2(t)) = (q_3(t), p_3(t), q_4(t), p_4(t))$, we find that S_1 reduces to just the standard phase-space action S' up to a surface term. Also, the imaginary part of the action S_2, reduces to a pure surface term using the classical equations of motion.

4.1.3 Constraint Hypersurface

We now find the states over which the matrix elements of the evolution operator may be evaluated in the propagator such that in the classical limit we remain on the constraint surface. Note that the constraints in our model are satisfied, at the classical level, if $q_1(t) - q_2(t) = 0$ and $p_1(t) - p_2(t) = 0$. For these equations to hold we must have

$$q_1(0) = q_2(0), \qquad p_1(T) = p_2(T),$$
$$\dot{q}_1(0) = \dot{q}_2(0), \qquad \dot{p}_1(T) = \dot{p}_2(T). \tag{4.32}$$

Using the definitions (28), the equations of motion (29), and the boundary conditions (24) we find that the conditions (32) for the constraints to hold imply that $\alpha' = \beta'$ and $\alpha'' = \beta''$. Thus, in our propagator the states $|\psi\rangle = |\alpha', \beta'\rangle$ or $|\alpha'', \beta''\rangle$ are not arbitrary but must be restricted as discussed above and so the allowed states are $|\psi\rangle = |\alpha', \alpha'\rangle$ or $|\alpha'', \alpha''\rangle$. Also, for these allowed states we find that $\langle\psi| : \sigma_i : |\psi\rangle = 0$, i.e. the expectation values of the constraints as operators vanish.

4.2 Toy Model 2

Here we study a second model with four second-class constraints described by the Lagrangian

$$L(\mathbf{x}, \dot{\mathbf{x}}, y, \dot{y}) = \frac{1}{2}\dot{\mathbf{x}}^2 - y(\mathbf{x}^2 - R^2) \tag{4.33}$$

where $\mathbf{x} = (x_1, x_2)$, a two dimensional vector, and y are dynamical variables. As a first step toward quantization we go over to the Hamiltonian formalism. The canonically conjugate momenta to the coordinates are

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}}, \qquad \pi = \frac{\partial L}{\partial \dot{y}} = 0. \tag{4.34}$$

Therefore, the canonical Hamiltonian is given by

$$H = \frac{1}{2}\mathbf{p}^2 + y(\mathbf{x}^2 - R^2). \tag{4.35}$$

Thus, we have a system with one primary constraint $\pi = 0$. We of course want our primary constraint to hold at all times, so, we require

$$\dot{\pi} = \{\pi, H\} = (\mathbf{x}^2 - R^2) = \sigma_1 = 0, \tag{4.36}$$

i.e., we have a secondary constraint $\sigma_1 = 0$. We find that our system has only two further constraints; $\dot{\sigma}_1 = \{\sigma_1, H\} = 2\mathbf{x}\mathbf{p} = 2\sigma_2 = 0$ and $\dot{\sigma}_2 = \{\sigma_2, H\} = \mathbf{p}^2 - 2y\mathbf{x}^2 = \sigma_3 = 0$. It can be easily verified that the four constraints

$$\pi = 0, \qquad \sigma_1 = (\mathbf{x}^2 - R^2) = 0, \qquad \sigma_2 = \mathbf{x}\mathbf{p} = 0, \qquad \sigma_3 = \mathbf{p}^2 - 2y\mathbf{x}^2 = 0, \tag{4.37}$$

form a set of second-class constraints. We use H_{ph} to identify the physical subspace in the Hilbert space under consideration. Since the physical Hamiltonian is $H_{ph} = (H)_{\pi=0,\sigma_i=0}$, we find that

$$H_{ph} = \frac{(\mathbf{p}T\mathbf{x})^2}{2R^2} \tag{4.38}$$

where $\mathbf{p}T\mathbf{x} = (p_1 x_2 - p_2 x_1)$ and in obtaining H_{ph} we have used the fact that $\mathbf{p}^2$ can be written as

$$\mathbf{p}^2 = \frac{(\mathbf{p}\mathbf{x})^2}{\mathbf{x}^2} + \frac{(\mathbf{p}T\mathbf{x})^2}{\mathbf{x}^2}. \tag{4.39}$$

We see that H_{ph} is proportional to the square of the z-component of the orbital angular momentum and its eigenstates are known. Now, let us promote $\mathbf{x}, \mathbf{p}, y, \pi$ to operators and express them using creation and annihilation operators as

$$\begin{aligned}
x_j &= \frac{(a_j + a_j^\dagger)}{\sqrt{2}}, & p_j &= \frac{(a_j - a_j^\dagger)}{\sqrt{2}i}, & j &= 1,2 \\
y &= \frac{(a_3 + a_3^\dagger)}{\sqrt{2}}, & \pi &= \frac{(a_3 - a_3^\dagger)}{\sqrt{2}i},
\end{aligned} \tag{4.40}$$

where $[a_i, a_j^\dagger] = \delta_{ij}$ and $[a_i, a_j] = 0$ for $i, j = 1, 2, 3$. Using the definitions in (40) we

find that $x_1 p_2 - x_2 p_1 = i(a_2^\dagger a_1 - a_1^\dagger a_2)$. Transforming to new operators defined by

$$A_1 = \frac{(a_1 - ia_2)}{\sqrt{2}}, \qquad A_2 = \frac{(a_1 + ia_2)}{\sqrt{2}}, \qquad (4.41)$$

we can write $x_1 p_2 - x_2 p_1 = i(a_2^\dagger a_1 - a_1^\dagger a_2) = (A_1^\dagger A_1 - A_2^\dagger A_2)$, where the operators

A_i satisfy the following commutation relations; $[A_i, A_j^\dagger] = \delta_{ij}$ and $[A_i, A_j] = 0$ for

$i, j = 1, 2$. The states $|l, m\rangle$ given by

$$|l, m\rangle = \frac{(A_1^\dagger)^{l+m}(A_2^\dagger)^{l-m}}{\sqrt{(l+m)!(l-m)!}}|0, 0\rangle, \qquad (4.42)$$

are eigenstates of $(A_1^\dagger A_1 - A_2^\dagger A_2)$, where $l = 1, 2, ...$ and $m = -l, -l+1, ..., l$ (see

Schwinger's oscillator model for angular momentum [22]). But recall that our problem

is being described using three dynamical degrees of freedom, namely (x,y), so, we write

the eigenstates $|l, m\rangle$ as follows

$$|l, m\rangle \equiv |l, m, 0\rangle = \frac{(a_1^\dagger + ia_2^\dagger)^{l+m}(a_1^\dagger - ia_2^\dagger)^{l-m}}{(\sqrt{2})^{l+m}(\sqrt{2})^{l-m}\sqrt{(l+m)!(l-m)!}}|0, 0, 0\rangle, \qquad (4.43)$$

where $a_3|l, m, 0\rangle = 0$.

4.2.1 Projection Operator

We saw above that the eigenstates of H_{ph} are the states $|l, m\rangle$ as written in (43).

Hence, a projection operator that will bring an arbitrary state from the Hilbert space

under consideration onto the physical subspace is given by

$$P = \sum_{m=-l}^{l} |l, m\rangle\langle l, m|. \qquad (4.44)$$

We will now show how to write the projection operator as an integral. We multiply P

by unity on either side,

$$\begin{aligned}
P &= \sum_m \int \frac{d^2\alpha d^2\beta d^2\eta}{\pi^3}|\alpha, \beta, \eta\rangle\langle\alpha, \beta, \eta|l, m\rangle\langle l, m| \int \frac{d^2\gamma d^2\delta d^2\xi}{\pi^3}|\gamma, \delta, \xi\rangle\langle\gamma, \delta, \xi| \\
&= \sum_m \int \frac{d^2\alpha d^2\beta d^2\eta d^2\gamma d^2\delta d^2\xi}{\pi^6}|\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi|\langle\alpha, \beta, \eta|l, m\rangle\langle l, m|\gamma, \delta, \xi\rangle.
\end{aligned} \qquad (4.45)$$

In the above expression the states $|\alpha, \beta, \eta\rangle$ and $|\gamma, \delta, \xi\rangle$ are eigenstates of annihilation operators a_i, for $i = 1, 2, 3$. Using the fact that for canonical coherent states $\langle z|0\rangle = \langle 0|z\rangle = \exp(-\frac{1}{2}|z|^2)$, and the definition in (43) for the states $|l, m\rangle$ we can write the projection operator as

$$
\begin{aligned}
P = \sum_m \int & \frac{d^2\alpha d^2\beta d^2\eta d^2\gamma d^2\delta d^2\xi}{\pi^6} e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2)} \times \\
& |\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi| \frac{(\alpha^\star + i\beta^\star)^{l+m}(\alpha^\star - i\beta^\star)^{l-m}}{2^l \sqrt{(l+m)!(l-m)!}} \frac{(\gamma - i\delta)^{l+m}(\gamma + i\delta)^{l+m}}{2^l \sqrt{(l+m)!(l-m)!}}.
\end{aligned}
\tag{4.46}
$$

The summation in the above expression can easily be done, so the projection operator is finally given by

$$
\begin{aligned}
P = \int & \frac{d^2\alpha d^2\beta d^2\eta d^2\gamma d^2\delta d^2\xi}{\pi^6} |\alpha, \beta, \eta\rangle\langle\gamma, \delta, \xi| \frac{(\alpha^\star\gamma + \beta^\star\delta)^{2l}}{2l!} \\
& \times e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + |\eta|^2 + |\gamma|^2 + |\delta|^2 + |\xi|^2)},
\end{aligned}
\tag{4.47}
$$

a weighted integral over bicoherent states.

4.2.2 Path Integral

We are now ready to construct a path integral representation of the canonical coherent state propagator using our assertion that for a constrained system one should use an appropriate projection operator rather than the resolution of unity at every time slice. Thus, the propagator is given by

$$
\begin{aligned}
\langle\alpha'', \beta'', \eta''|e^{-iT\mathcal{H}}|\alpha', \beta', \eta'\rangle = & \\
\langle\alpha'', \beta'', \eta''|e^{-i\epsilon\mathcal{H}} P_N e^{-i\epsilon\mathcal{H}} & ... P_1 e^{-i\epsilon\mathcal{H}}|\alpha', \beta', \eta'\rangle,
\end{aligned}
\tag{4.48}
$$

where in the above expression $\mathcal{H}$, the normal ordered Hamiltonian is

$$
\begin{aligned}
\mathcal{H} =: \frac{1}{2}\mathbf{p}^2 + y(\mathbf{x}^2 - R^2) := & \; : \frac{1}{2}[\frac{(a_3 + a_3^\dagger)}{\sqrt{2}} - \frac{1}{2}][a_1^2 + a_1^{\dagger 2} + a_2^2 + a_2^{\dagger 2}] \\
& + [\frac{(a_3 + a_3^\dagger)}{\sqrt{2}} + \frac{1}{2}](a_1^\dagger a_1 + a_2^\dagger a_2) - R^2 \frac{(a_3 + a_3^\dagger)}{\sqrt{2}} : .
\end{aligned}
\tag{4.49}
$$

In (48), P_n, the projection operator at the nth time slice, is

$$
\begin{aligned}
P_n = \int & \frac{d^2\alpha_n d^2\beta_n d^2\eta_n d^2\gamma_n d^2\delta_n d^2\xi_n}{\pi^6} |\alpha_n, \beta_n, \eta_n\rangle\langle\gamma_n, \delta_n, \xi_n| \\
& \times \frac{(\alpha_n^\star\gamma_n + \beta_n^\star\delta_n)^{2l}}{2l!} e^{-\frac{1}{2}(|\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\gamma_n|^2 + |\delta_n|^2 + |\xi_n|^2)}.
\end{aligned}
\tag{4.50}
$$

As in section 4.1 we can write the propagator formally as an integral over continuous and differentiable paths which is given by

$$\langle \alpha'', \beta'', \eta'' | e^{-iT\mathcal{H}} | \alpha', \beta', \eta' \rangle = \int e^{i\int_0^T \{\frac{i}{2}[\gamma^\star\dot\alpha - \dot\gamma^\star\alpha + \delta^\star\dot\beta - \dot\delta^\star\beta + \xi^\star\dot\eta - \dot\xi^\star\eta] - \langle\mathcal{H}\rangle\}dt} \mathcal{D}\mu. \quad (4.51)$$

In the expression above

$$\langle \mathcal{H} \rangle = \frac{1}{2}[\frac{(\xi^\star + \eta)}{\sqrt{2}} - \frac{1}{2}][\alpha^2 + \gamma^{\star 2} + \beta^2 + \delta^{\star 2}]$$
$$+ [\frac{(\xi^\star + \eta)}{\sqrt{2}} + \frac{1}{2}](\gamma^\star\alpha + \delta^\star\beta) - R^2\frac{(\xi^\star + \eta)}{\sqrt{2}}, \quad (4.52)$$

the discrete form of the measure is

$$\mathcal{D}\mu = \prod_n \{\frac{d^2\alpha_n d^2\beta_n d^2\eta_n d^2\gamma_n d^2\delta_n d^2\xi_n}{\pi^6} \frac{(\alpha_n^\star\gamma_n + \beta_n^\star\delta_n)^{2l}}{2l!}$$
$$\times e^{-(|\alpha_n|^2 + |\beta_n|^2 + |\eta_n|^2 + |\gamma_n|^2 + |\delta_n|^2 + |\xi_n|^2) + \gamma_n^\star\alpha_n + \delta_n^\star\beta_n + \xi_n^\star\eta_n}\}, \quad (4.53)$$

and the boundary conditions are

$$(\alpha(0), \beta(0), \eta(0)) = (\gamma(0), \delta(0), \xi(0)) = (\alpha', \beta', \eta')$$

$$(\alpha(T), \beta(T), \eta(T)) = (\gamma(T), \delta(T), \xi(T)) = (\alpha'', \beta'', \eta''). \quad (4.54)$$

4.2.3 Classical Limit

The phase-space action according to the propagator in (51), up to a surface term, is given by

$$S = \int_0^T \{i[\gamma^\star\dot\alpha + \delta^\star\dot\beta + \xi^\star\dot\eta] - \frac{1}{2}[\frac{(\xi^\star + \eta)}{\sqrt{2}} - \frac{1}{2}][\alpha^2 + \gamma^{\star 2} + \beta^2 + \delta^{\star 2}]$$
$$- [\frac{(\xi^\star + \eta)}{\sqrt{2}} + \frac{1}{2}](\gamma^\star\alpha + \delta^\star\beta) + R^2\frac{(\xi^\star + \eta)}{\sqrt{2}}\}dt. \quad (4.55)$$

Extremising this action, while keeping the end points of the paths in them fixed, we obtain the following equations of motion

$$\dot\alpha = \frac{(\gamma^\star + \alpha)(\xi^\star + \eta)}{\sqrt{2}i} - \frac{(\gamma^\star - \alpha)}{2i}, \qquad \dot\gamma = \frac{(\alpha^\star + \gamma)(\xi + \eta^\star)}{\sqrt{2}i} - \frac{(\alpha^\star - \gamma)}{2i},$$

$$\dot\beta = \frac{(\delta^\star + \beta)(\xi^\star + \eta)}{\sqrt{2}i} - \frac{(\delta^\star - \beta)}{2i}, \qquad \dot\delta = \frac{(\beta^\star + \delta)(\xi + \eta^\star)}{\sqrt{2}i} - \frac{(\beta^\star - \delta)}{2i},$$

$$\dot\eta = \frac{1}{2\sqrt{2}i}[\alpha^2 + \gamma^{\star 2} + \beta^2 + \delta^{\star 2}] + \frac{(\gamma^\star\alpha + \delta^\star\beta - R^2)}{\sqrt{2}i},$$

$$\dot\xi = \frac{1}{2\sqrt{2}i}[\alpha^{\star 2} + \gamma^2 + \beta^{\star 2} + \delta^2] + \frac{(\gamma\alpha^\star + \delta\beta^\star - R^2)}{\sqrt{2}i}. \qquad (4.56)$$

Using these equations of motion and the boundary conditions (54) one finds that $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$ on the classical trajectories. Next, introducing the following definitions

$$\alpha = \frac{(q_1 + ip_1)}{\sqrt{2}}, \qquad \beta = \frac{(q_2 + ip_2)}{\sqrt{2}}, \qquad \eta = \frac{(q_3 + ip_3)}{\sqrt{2}},$$
$$\gamma = \frac{(q_4 + ip_4)}{\sqrt{2}}, \qquad \delta = \frac{(q_5 + ip_5)}{\sqrt{2}}, \qquad \xi = \frac{(q_6 + ip_6)}{\sqrt{2}}, \qquad (4.57)$$

and using the fact that the equality $(\alpha(t), \beta(t), \eta(t)) = (\gamma(t), \delta(t), \xi(t))$ implies $(q_1(t), p_1(t), q_2(t), p_2(t), q_3(t), p_3(t)) = (q_4(t), p_4(t), q_5(t), p_5(t), q_6(t), p_6(t))$ we can write the equations of motion (56) in terms of the variables $(q_1, p_1, q_2, p_2, q_3, p_3)$ as

$$\dot{q}_1 = p_1, \qquad \dot{p}_1 = -2q_1 q_3, \qquad \dot{q}_2 = p_2,$$
$$\dot{p}_2 = -2q_2 q_3, \qquad \dot{q}_3 = 0, \qquad \dot{p}_3 = R^2 - (q_1^2 + q_2^2). \qquad (4.58)$$

These are precisely the equations one would obtain from the standard classical phase-space action

$$S' = \int_0^T \{[k_1 \dot{x}_1 + k_2 \dot{x}_2 + k_3 \dot{x}_3] - [\frac{1}{2}(k_1^2 + k_2^2) + x_3(x_1^2 + x_2^2 - R^2)]\} dt \qquad (4.59)$$

if we make the identification $(q_1, p_1, q_2, p_2, q_3, p_3) \leftrightarrow (x_1, k_1, x_2, k_2, x_3, k_3)$. Thus, we see our quantization procedure has again given us the correct classical limit.

4.2.4 Constraint Hypersurface

Here, as we did for model 1, we determine the restrictions on the states $|\alpha', \beta', \eta'\rangle$ and $|\alpha'', \beta'', \eta''\rangle$ over which the matrix element of the evolution operator can be evaluated so that we remain on the constraint surface in the classical limit. Recall the constraints (37) and the equations of motion (58); using the boundary conditions (54) we see that $\pi = p_3 = 0$ implies $\eta_I' = \eta_I'' = 0$. Also, $\dot{q}_3 = 0$ implies $q_3 = c = \sqrt{2}\eta_R' = \sqrt{2}\eta_R''$, where c is a real constant. Here (η_R, η_I) are the real and imaginary parts of η respectively. Next, the equation $\dot{p}_3 = -\sigma_1 = R^2 - (q_1^2 + q_2^2) = 0$ implies

$R^2 - 2\left(\alpha_R'^2 + \beta_R'^2\right) = R^2 - 2\left(\alpha_R''^2 + \beta_R''^2\right) = 0$. Also, the equation $\dot{\sigma}_1 = 2(q_1\dot{q}_1 + q_2\dot{q}_2) = 0$, requires that the following be true, $\alpha_R'\alpha_I' + \beta_R'\beta_I' = \alpha_R''\alpha_I'' + \beta_R''\beta_I'' = 0$. The last constraint $\sigma_3 = (p_1^2 + p_2^2) - 2q_3(q_1^2 + q_2^2) = 0$, implies $(\alpha_I'^2 + \beta_I'^2) - 2c(\alpha_R'^2 + \beta_R'^2) = (\alpha_I''^2 + \beta_I''^2) - 2c(\alpha_R''^2 + \beta_R''^2) = 0$. Thus, in the propagator the states $|\alpha', \beta', \eta'\rangle$ and $|\alpha'', \beta'', \eta''\rangle$ are not arbitrary but must be restricted as noted above. We summarize the restrictions on the state $|\alpha', \beta', \eta'\rangle$ below for convenience,

$$R^2 - 2\left(\alpha_R'^2 + \beta_R'^2\right) = 0, \qquad\qquad \alpha_R'\alpha_I' + \beta_R'\beta_I' = 0,$$

$$(\alpha_I'^2 + \beta_I'^2) - 2c(\alpha_R'^2 + \beta_R'^2) = 0, \qquad \eta' = \frac{c}{\sqrt{2}}. \tag{4.60}$$

With these restrictions on the states $|\psi\rangle = |\alpha', \beta', \eta'\rangle$ and $|\alpha'', \beta'', \eta''\rangle$, we find that $\langle\psi| : \pi : |\psi\rangle = 0$ and $\langle\psi| : \sigma_i : |\psi\rangle = 0$ for the constraints in (37).

CHAPTER 5
E(2) COHERENT STATES

In this chapter we consider a system with a Holonomic constraint. We will show how to construct path integrals for such systems. The formalism described here relies on the ability to be able identify the appropriate group for the specific system under consideration. The problem we consider is that of quantizing a system whose configuration space is a circle, hence, the Holonomic constraint can be expressed as $(x_1^2 + x_2^2 - R^2) = 0$. We will construct the Universal propagator for such a system. The universal propagator is such that it correctly propagates coherent-state representatives of state vectors independent of the fiducial vector used in the construction of the coherent states. A universal propagator was first introduced for the Heisenberg-Weyl group by Klauder [23]. Subsequently the universal propagator has been written for the affine or $ax + b$ group and for the SU(2) group [24,25].

For the quantization of systems whose configuration space is the (n-1) sphere S^{n-1}, the canonical group that arises naturally is the n-dimensional Euclidean group E(n)-the semidirect product $\mathbb{R}^n \circledS SO(n)$ of the abelian group $\mathbb{R}^n$ of translations in n-dimensions with the group SO(n) of rotations in n-dimensions [26]. In addition to the fact that a system whose configuration space is the (n-1) sphere has a single Holonomic constraint, a theory of coherent states for E(n) could throw new light on the much discussed question of the precise definition of a path integral for a system whose configuration space is a curved manifold.

We first review the construction of the universal propagator for the canonical coherent states [23]. This will set the stage for the construction of the universal propagator for the E(2) coherent states. Let P and Q denote an irreducible pair of self-adjoint Heisenberg

operators satisfying $[Q, P] = i$, and let

$$|p, q, \eta\rangle = e^{-iqP} e^{ipQ} |\eta\rangle \tag{5.1}$$

denote a family of normalized states for a fixed fiducial vector $|\eta\rangle$ with $\langle \eta | \eta \rangle = 1$, and $(p, q) \in \mathbb{R}^2$. These states are canonical coherent states and they admit a resolution of unity in the form,

$$\int |p, q, \eta\rangle \langle p, q, \eta| \frac{dp \, dq}{2\pi} = I, \tag{5.2}$$

for any $|\eta\rangle$ when integrated over all phase space [18]. These states lead to a representation of Hilbert space H by bounded, continuous functions,

$$\psi_\eta(p, q) = \langle p, q, \eta | \psi \rangle \tag{5.3}$$

defined for all $|\psi\rangle \in$ H. This representation is evidently dependent on the fiducial vector $|\eta\rangle$. An inner product in this representation is given by

$$\langle \varphi | \psi \rangle = \int \varphi_\eta^\star(p, q) \psi_\eta(p, q) \frac{dp \, dq}{2\pi}, \tag{5.4}$$

an integral which is independent of the fiducial vector $|\eta\rangle$. The Schrödinger equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \tag{5.5}$$

and its formal solution in terms of the evolution operator $U(t) = e^{-itH}$

$$|\psi(t'')\rangle = U(t'' - t') |\psi(t')\rangle, \tag{5.6}$$

are given in this representation, respectively, by

$$i \frac{\partial}{\partial t} \psi_\eta(p, q, t) = \langle p, q, \eta | H(P, Q) | \psi(t) \rangle \tag{5.7}$$

and

$$\psi_\eta(p'', q'', t'') = \int K_\eta(p'', q'', t''; p', q', t') \psi_\eta(p', q', t') \frac{dp' \, dq'}{2\pi}. \tag{5.8}$$

The integral kernel in the expression above is

$$K_\eta(p'', q'', t''; p', q', t') = \langle p'', q'', \eta | e^{-i(t''-t')H} | p', q', \eta \rangle. \tag{5.9}$$

Clearly K_η depends on the choice of the fiducial vector $|\eta\rangle$. In contrast, the universal propagator $K(p'', q'', t''; p', q', t')$ is a single function, independent of any particular fiducial vector, which nevertheless, propagates each of the ψ_η correctly, i.e.,

$$\psi_\eta(p'', q'', t'') = \int K(p'', q'', t''; p', q', t') \psi_\eta(p', q', t') \frac{dp' dq'}{2\pi} \tag{5.10}$$

for any $|\eta\rangle$. This function is constructed in two steps. First, one observes that

$$\begin{aligned}
(-i\frac{\partial}{\partial q})\psi_\eta(p, q) &= \langle p, q, \eta | P | \psi \rangle, \\
(q + i\frac{\partial}{\partial p})\psi_\eta(p, q) &= \langle p, q, \eta | Q | \psi \rangle,
\end{aligned} \tag{5.11}$$

independently of $|\eta\rangle$. Thus, if $H = H(P, Q)$ denotes a Hamiltonian, it follows that Schrödinger's equation takes the form

$$\begin{aligned}
i\frac{\partial}{\partial t}\psi_\eta(p, q, t) &= \langle p, q, t | H(P, Q) | \psi \rangle \\
&= H(-i\frac{\partial}{\partial q}, q + i\frac{\partial}{\partial p})\psi_\eta(p, q, t)
\end{aligned} \tag{5.12}$$

valid for all $|\eta\rangle$. We note that the universal propagator is also a solution to Schrödinger's equation

$$i\frac{\partial}{\partial t}K(p, q, t; p', q', t') = H(-i\frac{\partial}{\partial q}, q + i\frac{\partial}{\partial p})K(p, q, t; p', q', t'). \tag{5.13}$$

Second, one interprets the resulting Schrödinger's equation as an equation for *two* canonical degrees of freedom. In this interpretation $y_1 = q$ and $y_2 = p$ are viewed as *two* "coordinates", and one is looking at the irreducible Schrödinger representation of a special class of two-variable Hamiltonians, ones where the classical Hamiltonian is restricted to have the form $H_c(p_1, y_1 - p_2)$. Based on this interpretation a standard phase-space path integral solution may be given for the universal propagator between

sharp Schrödinger states. In particular, it follows, after some change of variables, that the universal propagator is given by the formal path integral

$$K(p'', q'', t''; p', q', t') = \int e^{i \int \{q\dot{p} - k\dot{q} - x\dot{p} - H(k,x)\} dt} \mathcal{D}p \mathcal{D}q \mathcal{D}k \mathcal{D}x, \qquad (5.14)$$

where x and k are the "momenta" conjugate to the "coordinates" p and q [23].

We will now show that it is possible to introduce an appropriate universal propagator for E(2) coherent states by following the construction outlined above.

5.1 E(2) Coherent States and their Propagators

5.1.1 E(2) Coherent State Representative of Functions on $L^2(S^1, d\theta)$

If a group G acts linearly on a vector space V, the group law on the semidirect product V⑤G is defined to be

$$(v_2, g_2)(v_1, g_1) = (v_2 + R(g_2)v_1, g_2 g_1) \qquad (5.15)$$

where R(g) is the operator representing $g \in G$ by its linear action on V. In the case of $E(2) = \mathbb{R}^2 ⑤ SO(2)$, a unitary representation of the group is associated with the commutation relations of the self-adjoint operators

$$[X, Y] = 0, \quad [X, J] = -iY, \quad [Y, J] = iX, \qquad (5.16)$$

where X and Y are two translation generators and J is the generator of rotations [27]. For the group E(2) there is a one-parameter family of unitarily inequivalent irreducible representations, each of which can be realized on the Hilbert space $L^2(S^1, d\theta)$ of square integrable, periodic functions of an angle variable θ with $-\pi \le \theta < \pi$,

$$(X\psi)(\theta) = r\cos\theta\psi(\theta), \quad (Y\psi)(\theta) = r\sin\theta\psi(\theta), \quad (J\psi)(\theta) = -i\frac{\partial\psi}{\partial\theta}(\theta). \qquad (5.17)$$

The representation of the group itself can be taken as

$$(U(a, b, c)\psi)(\theta) = e^{-iar\cos\theta - ibr\sin\theta}\psi(\theta - c), \qquad (5.18)$$

where (a, b) and c are elements of $\mathbb{R}^2$ and SO(2), respectively. The parameter r that specifies the representation lies in the range $0 < r < \infty$ and its square is the value of the Casimir operator $X^2 + Y^2$. We are going to consider those reducible representations that can be written as a direct integral of irreducible representations with the parameter r varying in the range $0 < R \le r < \overline{R} < \infty$. The simplest way of realizing such a direct integral is to use the Hilbert space $\overline{L}^2$ of functions defined on the annulus $0 < R \le r < \overline{R} < \infty$, in the plane. So, with each fiducial vector $|\eta\rangle \in \overline{L}^2$, we associate the family of coherent states,

$$|a, b, c, \eta\rangle = e^{iaR} e^{-iaX} e^{-ibY} e^{-icJ} |\eta\rangle, \tag{5.19}$$

where the fiducial vector satisfies

$$\langle \eta | \eta \rangle = \int\limits_{R}^{\overline{R}} \int\limits_{S^1} |\eta(r, \theta)|^2 r \, dr \, d\theta = 1. \tag{5.20}$$

The factor e^{iaR} has been appended to the group action to ensure proper limiting behavior as $R \to \infty$. The map from vectors $\psi \in L^2(S^1, d\theta)$ to functions $\psi_\eta(a, b, c)$ on E(2) is defined by

$$\psi_\eta(a, b, c) = \langle a, b, c, \eta | \psi \rangle = \int\limits_{R}^{\overline{R}} \int\limits_{S^1} \langle a, b, c, \eta | r, \theta \rangle \langle r, \theta | \psi \rangle r \, dr \, d\theta$$

$$= e^{-iaR} \left[\frac{2}{(\overline{R}^2 - R^2)} \right]^{\frac{1}{2}} \int\limits_{R}^{\overline{R}} \int\limits_{S^1} \eta^\star(r, \theta - c) e^{iar\cos\theta + ibr\sin\theta} \psi(\theta) r \, dr \, d\theta, \tag{5.21}$$

where we have used the resolution of unity

$$\int\limits_{R}^{\overline{R}} \int\limits_{S^1} |r, \theta\rangle \langle r, \theta| r \, dr \, d\theta = I \tag{5.22}$$

in terms of the delta normalized eigenkets $|r, \theta\rangle$. Also, since $\psi \in L^2(S^1, d\theta)$, we must have

$$\langle r, \theta | \psi \rangle = \left[\frac{2}{(\overline{R}^2 - R^2)} \right]^{\frac{1}{2}} \psi(\theta) \tag{5.23}$$

to ensure that

$$\langle\psi|\psi\rangle = \int\limits_{R}^{\overline{R}} \int\limits_{S^1} \langle\psi|r,\theta\rangle\langle r,\theta|\psi\rangle r dr d\theta$$

(5.24)

$$= \frac{2}{\overline{R}^2 - R^2} \int\limits_{R}^{\overline{R}} \int\limits_{S^1} |\psi(\theta)|^2 r dr d\theta = \int\limits_{S^1} |\psi(\theta)|^2 d\theta.$$

Thus, the 'length' of a vector is

$$\langle\psi|\psi\rangle = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\pi}^{\pi} \langle\psi|a,b,c,\eta\rangle\langle a,b,c,\eta|\psi\rangle \frac{(\overline{R}^2 - R^2)}{2} \frac{da db dc}{(2\pi)^2}$$

$$= \int |e^{-iaR}[\frac{2}{(\overline{R}^2 - R^2)}]^{\frac{1}{2}} \int \eta^\star(r,\theta - c)e^{iar\cos\theta + ibr\sin\theta}\psi(\theta)r dr d\theta|^2 d\mu(a,b,c) \quad (5.25)$$

$$= \int\limits_{S^1} |\psi(\theta)|^2 d\theta.$$

In the expression above we have set $d\mu(a,b,c) = [(\overline{R}^2 - R^2)/2][da db dc/(2\pi)^2]$, and used the fact that

$$\int\limits_{R}^{\overline{R}} \int\limits_{-\pi}^{\pi} |\eta(r,\theta - c)|^2 r dr dc = \int\limits_{R}^{\overline{R}} \int\limits_{-\pi}^{\pi} |\eta(r,c)|^2 r dr dc = 1.$$

(5.26)

The functions $\psi_\eta(a,b,c)$ are bounded and continuous, and the set of functions on E(2) obtained by this transformation on $\psi \in L^2(S^1, d\theta)$ is a proper subspace, say $\mathcal{C}$, of all elements of $L^2(\mathbb{R}^2 \times S^1, d\mu(a,b,c))$. An inverse transform from $\psi_\eta(a,b,c) \in \mathcal{C}$, to $\psi \in L^2(S^1, d\theta)$ is given by

$$[\frac{2}{(\overline{R}^2 - R^2)}]^{\frac{1}{2}} \int\limits_{R}^{\overline{R}} r dr e^{iaR} \int e^{-iar\cos\theta - ibr\sin\theta}\eta(r,\theta - c)\psi_\eta(a,b,c)d\mu(a,b,c)$$

$$= [\frac{2}{(\overline{R}^2 - R^2)}]^{\frac{1}{2}} \int\limits_{R}^{\overline{R}} r dr \int \langle r,\theta|a,b,c,\eta\rangle\langle a,b,c,\eta|\psi\rangle d\mu(a,b,c)$$

(5.27)

$$= [\frac{2}{(\overline{R}^2 - R^2)}]^{\frac{1}{2}} \int\limits_{R}^{\overline{R}} r dr \langle r,\theta|\psi\rangle = \psi(\theta).$$

An inner product is given by

$$\langle\phi|\psi\rangle = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\pi}^{\pi} \phi_\eta^\star(a,b,c)\psi_\eta(a,b,c)d\mu(a,b,c)$$

$$= \int\limits_{S^1} \phi^\star(\theta)\psi(\theta)d\theta. \tag{5.28}$$

These properties demonstrate the unitarity of the coherent-state transformation between $L^2(S^1, d\theta)$ and $\mathcal{C}$. Since $\mathcal{C} \subset L^2(\mathbb{R}^2 \times S^1, d\mu(a,b,c))$ is isomorphic to $L^2(S^1, d\theta)$, the functions $\psi(\theta) = \langle\theta|\psi\rangle$ and $\psi_\eta(a,b,c) = \langle a,b,c,\eta|\psi\rangle$ can be regarded as two different representations of the single abstract ket $|\psi\rangle$. Thus, in the bra-ket notation, the basic result (25) can be written in the form

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\pi}^{\pi} |a,b,c,\eta\rangle\langle a,b,c,\eta|d\mu(a,b,c) = I, \tag{5.29}$$

which is the usual form of the coherent-state resolution of unity, albeit for a reducible representation of E(2).

5.1.2 Surface Constant Fiducial Vectors

To make full use of coherent states, we wish to have available the reproducing kernel property

$$\langle a'',b'',c'',\eta|\psi\rangle = \int \langle a'',b'',c'',\eta|a',b',c',\eta\rangle\langle a',b',c',\eta|\psi\rangle d\mu(a',b',c') \tag{5.30}$$

for arbitrary $\psi \in L^2(S^1, d\theta)$, and the idempotent property

$$\langle a'',b'',c'',\eta|a,b,c,\eta\rangle = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\pi}^{\pi} \langle a'',b'',c'',\eta|a',b',c',\eta\rangle$$

$$\times \langle a',b',c',\eta|a,b,c,\eta\rangle d\mu(a',b',c',). \tag{5.31}$$

The inner products in equations (30) and (31) do not involve vectors that lie in $L^2(S^1, d\theta)$ alone, simply because $\langle r,\theta|a,b,c,\eta\rangle$ depends on r. Consequently equations (30) and (31)

do not hold for a general fiducial vector $|\eta\rangle$. The restricted class of vectors for which the above two relations hold can be seen as follows. First, we have

$$\langle a'', b'', c'', \eta|\psi\rangle = e^{-ia''R}\left[\frac{2}{(\overline{R}^2 - R^2)}\right]^{\frac{1}{2}}$$

$$\times \int\limits_{R}^{\overline{R}} \int\limits_{S^1} \eta^\star(r, \theta - c'')e^{ia''r\cos\theta + ib''r\sin\theta}\psi(\theta)r\,dr\,d\theta. \tag{5.32}$$

Second, we also desire to have

$$\langle a'', b'', c'', \eta|\psi\rangle = \int \langle a'', b'', c'', \eta|a', b', c', \eta\rangle\langle a', b', c', \eta|\psi\rangle d\mu(a', b', c',)$$

$$= e^{-ia''R} \int \left\{ \int e^{i(a''-a')r\cos\theta + i(b''-b')r\sin\theta}\eta^\star(r, \theta - c'')\eta(r, \theta - c')r\,dr\,d\theta \right\} \times$$

$$\left\{\left[\frac{2}{(\overline{R}^2 - R^2)}\right]^{\frac{1}{2}} \int\limits_{R}^{\overline{R}} \int\limits_{S^1} e^{ia'\rho\cos\phi + ib'\rho\sin\phi}\eta^\star(\rho, \phi - c')\psi(\phi)\rho\,d\rho\,d\phi\right\}d\mu(a', b', c') \tag{5.33}$$

$$= e^{-ia''R}\left[\frac{2}{(\overline{R}^2 - R^2)}\right]^{\frac{1}{2}} \int\limits_{R}^{\overline{R}} \int\limits_{S^1} e^{ia''r\cos\theta + ib''r\sin\theta}\eta^\star(r, \theta - c'')\psi(\theta) \times$$

$$\left\{\int\limits_{-\pi}^{\pi} |\eta(r, \theta - c')|^2\frac{(\overline{R}^2 - R^2)}{2}dc'\right\}r\,dr\,d\theta.$$

Hence, for (33) to equal (32) we see that our fiducial vectors need to satisfy the condition,

$$\int\limits_{-\pi}^{\pi} |\eta(r, \theta - c')|^2 dc' = \int\limits_{-\pi}^{\pi} |\eta(r, c')|^2 dc' = \frac{2}{\overline{R}^2 - R^2}. \tag{5.34}$$

We will call such vectors "surface-constant" vectors. The conclusion is that if the reproducing kernel property (30) and the idempotency condition (31) are to hold, surface-constant fiducial vectors must be employed, even though the resolution of unity holds for more general fiducial vectors. All surface constant vectors can be written in the form

$$\eta(r, \theta) = \left[\frac{2}{(\overline{R}^2 - R^2)}\right]^{\frac{1}{2}}\frac{\xi(r, \theta)}{(\int\limits_{-\pi}^{\pi} |\xi(r, \alpha)|^2 d\alpha)^{\frac{1}{2}}} \tag{5.35}$$

for an arbitrary nonvanishing function $\xi(r, \theta)$, for almost all r and θ.

5.1.3 Propagators

The abstract Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle, \tag{5.36}$$

involving the self-adjoint Hamiltonian H, is formally solved with the aid of the evolution operator $U(t) = e^{-iHt}$, namely

$$|\psi(t'')\rangle = e^{-i(t''-t')H}|\psi(t')\rangle. \tag{5.37}$$

In the E(2) coherent-state representation the time evolution is affected by an integral kernel

$$K_\eta(a'',b'',c'',t'';a',b',c',t') = \langle a'',b'',c'',\eta|e^{-iH(t''-t')}|a',b',c',\eta\rangle \tag{5.38}$$

in the form

$$\psi_\eta(a'',b'',c'',t'') = \int K_\eta(a'',b'',c'',t'';a',b',c',t')\psi_\eta(a',b',c',t')d\mu(a',b',c'). \tag{5.39}$$

Clearly K_η depends strongly on the fiducial vector $|\eta\rangle$, as does ψ_η. The universal propagator $K(a'',b'',c'',t'';a',b',c',t')$ in contrast, is a single function independent of any fiducial vector, which, nevertheless, has the property that

$$\psi_\eta(a'',b'',c'',t'') = \int K(a'',b'',c'',t'';a',b',c',t')\psi_\eta(a',b',c',t')d\mu(a',b',c'), \tag{5.40}$$

holds just as before for any choice of fiducial vector $|\eta\rangle$. The functions K_η and K are qualitatively different as is clear from their behavior as $t'' \to t'$. In particular

$$\lim_{t'' \to t'} K_\eta(a'',b'',c'',t'';a',b',c',t') = \langle a'',b'',c'',\eta|a',b',c',\eta\rangle, \tag{5.41}$$

which clearly retains a strong dependence on the fiducial vector $|\eta\rangle$. On the other hand, if (40) is to hold for any $|\eta\rangle$ we must require that

$$\lim_{t'' \to t'} K(a'',b'',c'',t'';a',b',c',t') = \frac{2(2\pi)^2}{\overline{R}^2 - R^2}\delta(a''-a')\delta(b''-b')\delta(c''-c'). \tag{5.42}$$

Now let us turn our attention to a suitable differential equation satisfied by the propagators K_η and K. It is straightforward to see that

$$
\begin{aligned}
(-i\frac{\partial}{\partial a} + R)\langle a,b,c,\eta|\psi\rangle &= \langle a,b,c,\eta|X|\psi\rangle, \\
(-i\frac{\partial}{\partial b})\langle a,b,c,\eta|\psi\rangle &= \langle a,b,c,\eta|Y|\psi\rangle, \\
(-i\frac{\partial}{\partial c} - ia\frac{\partial}{\partial b} + ib\frac{\partial}{\partial a} - bR)\langle a,b,c,\eta|\psi\rangle &= \langle a,b,c,\eta|J|\psi\rangle,
\end{aligned}
\tag{5.43}
$$

hold quite independently of $|\eta\rangle$. Thus if $H = H(X,Y,J)$ denotes a Hamiltonian it follows that Schrödinger's equation takes the form

$$
\begin{aligned}
i\frac{\partial}{\partial t}\psi_\eta(a,b,c,t) &= \langle a,b,c,\eta|H(X,Y,J)|\psi(t)\rangle = \\
H(-i\frac{\partial}{\partial a} + R, -i\frac{\partial}{\partial b}, &-i\frac{\partial}{\partial c} - ia\frac{\partial}{\partial b} + ib\frac{\partial}{\partial a} - bR)\psi_\eta(a,b,c,t),
\end{aligned}
\tag{5.44}
$$

valid for any $|\eta\rangle$. The propagators are also solutions of Schrödingers equation so it follows that

$$
\begin{aligned}
i\frac{\partial}{\partial t}K_*(a,b,c,t;a',b',c',t') &= \\
H(-i\frac{\partial}{\partial a} + R, -i\frac{\partial}{\partial b}, -i\frac{\partial}{\partial c} &- ia\frac{\partial}{\partial b} + ib\frac{\partial}{\partial a} - bR)K_*(a,b,c,t;a',b',c',t'),
\end{aligned}
\tag{5.45}
$$

where K_* denotes either K_η or K. The initial conditions at $t = t'$, determines which function is under consideration.

Equation (45) admits two qualitatively different interpretations: when K_η is under consideration, the operators $(-i\frac{\partial}{\partial a} + R), (-i\frac{\partial}{\partial b})$ and $(-i\frac{\partial}{\partial c} - ia\frac{\partial}{\partial b} + ib\frac{\partial}{\partial a} - bR)$ refer to a single degree of freedom made irreducible by confining attention to the subspace $\mathcal{C}$ of $L^2(\mathbb{R}^2 \times S^1, d\mu(a,b,c))$ for a fixed $|\eta\rangle$ and for all $\psi \in L^2(S^1, d\theta)$. For K a different interpretation is appropriate.

5.2 The Universal Propagator

In contrast to the former case, when the universal propagator K is under consideration the resultant Schrödinger's equation (45) is interpreted as one appropriate to *three*

canonical degrees of freedom. In this view $x_1 = a$, $x_2 = b$, and $x_3 = c$ denote *three* "coordinates", and one is looking at the irreducible Schrödinger representation of a special class of Hamiltonians, ones where the classical Hamiltonian is restricted to the form $H_c(p_1 + R, p_2, p_3 + x_1 p_2 - x_2 p_1 - x_2 R)$, rather than the most general form $H_c(p_1, p_2, p_3, x_1, x_2, x_3)$. In the case of K, based on the interpretation described above, a standard phase-space path integral solution may be given for the universal propagator between sharp Schrödinger states. In particular it follows that

$$K(a'', b'', c'', t''; a', b', c', t') =$$
$$\int e^{i \int [p^a \dot{a} + p^b \dot{b} + p^c \dot{c} - H(p^a + R, p^b, p^c + a p^b - b p^a - b R)] dt} \mathcal{D}p^a \mathcal{D}p^b \mathcal{D}p^c \mathcal{D}a \mathcal{D}b \mathcal{D}c \tag{5.46}$$

Here p^a, p^b, and p^c are the "conjugate momenta" to the coordinates a, b, c, respectively. Note that the special form of the Hamiltonian has been used. Also, (p^a, p^b) are continuous, while p^c is a discrete variable. In the standard phase-space path integral there is always one more set of (p^a, p^b, p^c) integrations (summations) compared to the (a, b, c) family. This situation is made explicit in the regularized prescription for the path integral given, in standard notation, by

$$K(a'', b'', c'', t''; a', b', c', t') = \frac{2(2\pi)^2}{\overline{R}^2 - R^2} \lim_{\epsilon \to 0} \int \sum_{p^c_{N+\frac{1}{2}} = -\infty}^{\infty} \cdots \sum_{p^c_{\frac{1}{2}} = -\infty}^{\infty} \times$$
$$e^{i \sum_{l=0}^{N} [p^a_{l+\frac{1}{2}}(a_{l+1} - a_l) + p^b_{l+\frac{1}{2}}(b_{l+1} - b_l) + p^c_{l+\frac{1}{2}}(c_{l+1} - c_l) - \epsilon H]} \prod_{l=1}^{N} da_l db_l dc_l \prod_{l=0}^{N} \frac{dp^a_{l+\frac{1}{2}} dp^b_{l+\frac{1}{2}}}{(2\pi)^2}, \tag{5.47}$$

where in the expression above $(N + 1)\epsilon = (t'' - t')$ and the boundary conditions are $(a_{N+1}, b_{N+1}, c_{N+1}) = (a'', b'', c'')$ and $(a_0, b_0, c_0) = (a', b', c')$. Also,

$$H = H(p^a_{l+\frac{1}{2}} + R, p^b_{l+\frac{1}{2}}, p^c_{l+\frac{1}{2}} + \frac{(a_{l+1} + a_l)}{2} p^b_{l+\frac{1}{2}}$$
$$- \frac{(b_{l+1} + b_l)}{2} p^a_{l+\frac{1}{2}} - \frac{(b_{l+1} + b_l)}{2} R), \tag{5.48}$$

and the momenta $p^c = n$, where n is an integer. We will now construct several examples of the universal propagator.

5.2.1 Vanishing Hamiltonian

The propagator, when the Hamiltonian is zero, is given by

$$K(a'', b'', c'', t''; a', b', c', t') = \int e^{i \int [p^a \dot{a} + p^b \dot{b} + p^c \dot{c}] dt} \mathcal{D}p^a \mathcal{D}p^b \mathcal{D}p^c \mathcal{D}a \mathcal{D}b \mathcal{D}c$$

$$= \frac{2(2\pi)^2}{\overline{R}^2 - R^2} \int \sum_{p^c_{N+\frac{1}{2}}} \cdots \sum_{p^c_{\frac{1}{2}}} e^{i \sum_{l=0}^{N} [p^a_{l+\frac{1}{2}}(a_{l+1}-a_l) + p^b_{l+\frac{1}{2}}(b_{l+1}-b_l) + p^c_{l+\frac{1}{2}}(c_{l+1}-c_l)]}$$

$$\times \prod_{l=1}^{N} da_l db_l dc_l \prod_{l=0}^{N} \frac{dp^a_{l+\frac{1}{2}} dp^b_{l+\frac{1}{2}}}{(2\pi)^2} \tag{5.49}$$

$$= \frac{8\pi^2}{\overline{R}^2 - R^2} \delta(a'' - a') \delta(b'' - b') \int \prod_{l=0}^{N} \{\sum_{m_l} \delta(c_{l+1} - c_l + 2\pi m_l)\} \prod_{l=1}^{N} dc_l$$

$$= \frac{8\pi^2}{\overline{R}^2 - R^2} \delta(a'' - a') \delta(b'' - b') \delta(c'' - c').$$

In the above we have used the fact that $-\pi < c_l \le \pi$ and $m_l \in \mathbb{Z}$. Actually, with this restriction on c_l, only the $m_l = 0$ term contributes. It is evident we have obtained the correct result.

5.2.2 Linear Hamiltonian

Next we consider the linear Hamiltonian $H(X, Y, J) = \alpha X + \beta Y + \gamma J$. This case will help us in the proof of the fact that the universal propagator for a general Hamiltonian propagates all state vectors correctly. Thus, for the linear Hamiltonian

$$K(a'', b'', c'', t''; a', b', c', t') =$$

$$\int e^{i \int [p^a \dot{a} + p^b \dot{b} + p^c \dot{c} - \alpha(p^a + R) - \beta p^b - \gamma(p^c + ap^b - bp^a - bR)] dt} \mathcal{D}p^a \mathcal{D}p^b \mathcal{D}p^c \mathcal{D}a \mathcal{D}b \mathcal{D}c. \tag{5.50}$$

We now make the following change of variable $p^c \to p^c - ap^b + bp^a + bR$, and the

resultant regularized path integral becomes

$$
K(a'', b'', c'', t''; a', b', c', t') =
$$

$$
\int e^{i \int [p^a \dot{a} + p^b \dot{b} + p^c \dot{c} - ap^b \dot{c} + bp^a \dot{c} + bR\dot{c} - \alpha(p^a + R) - \beta p^b - \gamma p^c] dt} \mathcal{D}p^a \mathcal{D}p^b \mathcal{D}p^c \mathcal{D}a \mathcal{D}b \mathcal{D}c
$$

$$
= \frac{2(2\pi)^2}{\overline{R}^2 - R^2} e^{-i\alpha RT} \int \sum_{p^c_{N+\frac{1}{2}}} \cdots \sum_{p^c_{\frac{1}{2}}} \prod_{l=1}^{N} da_l db_l dc_l \prod_{l=0}^{N} \frac{dp^a_{l+\frac{1}{2}} dp^b_{l+\frac{1}{2}}}{(2\pi)^2} \times
$$

$$
e^{i \sum_{l=0}^{N} [p^a_{l+\frac{1}{2}} (a_{l+1} - a_l) + p^b_{l+\frac{1}{2}} (b_{l+1} - b_l) + p^c_{l+\frac{1}{2}} (c_{l+1} - c_l) - \frac{(a_{l+1} + a_l)}{2} p^b_{l+\frac{1}{2}} (c_{l+1} - c_l)]} \times
$$

$$
e^{i \sum_{l=0}^{N} [\frac{(b_{l+1} + b_l)}{2} p^a_{l+\frac{1}{2}} (c_{l+1} - c_l) + R \frac{(b_{l+1} + b_l)}{2} (c_{l+1} - c_l) - \epsilon \alpha p^a_{l+\frac{1}{2}} - \epsilon \beta p^b_{l+\frac{1}{2}} - \epsilon \gamma p^c_{l+\frac{1}{2}}]}
$$

$$
= \frac{8\pi^2 e^{-i\alpha RT}}{\overline{R}^2 - R^2} \delta(a'' - a' \cos \gamma T + b' \sin \gamma T - \frac{\alpha}{\gamma} \sin \gamma T - \frac{\beta}{\gamma}(\cos \gamma T - 1))
$$

$$
\times \delta(b'' - b' \cos \gamma T - a' \sin \gamma T - \frac{\beta}{\gamma} \sin \gamma T + \frac{\alpha}{\gamma}(\cos \gamma T - 1))
$$

$$
\times \sum_{m=-\infty}^{\infty} \{\delta(c'' - c' - \gamma T + 2\pi m)\}.
$$

(5.51)

Here we have set $(t'' - t') = T$. It can easily be verified that the above propagator satisfies the 'folding' property

$$
\int K(a'', b'', c'', T + S; a', b', c', S) K(a', b', c', S; a, b, c, 0) d\mu(a', b', c')
$$

$$
= K(a'', b'', c'', T + S; a, b, c, 0).
$$

(5.52)

5.2.3 A Quadratic Hamiltonian

The last example we consider here is the quadratic Hamiltonian $H(X, Y, J) = \alpha J^2$.

For this Hamiltonian the propagator in the regularized form is given by
$$K(a'', b'', c'', t''; a', b', c', t') =$$

$$\int e^{i \int [p^a \dot{a} + p^b \dot{b} + p^c \dot{c} - \alpha (p^c + a p^b - b p^a - bR)^2] dt} \mathcal{D}p^a \mathcal{D}p^b \mathcal{D}p^c \mathcal{D}a \mathcal{D}b \mathcal{D}c$$

$$= \int \sum_{p^c_{N+\frac{1}{2}}} \cdots \sum_{p^c_{\frac{1}{2}}} \prod_{l=1}^{N} da_l db_l dc_l \prod_{l=0}^{N} \frac{dp^a_{l+\frac{1}{2}} dp^b_{l+\frac{1}{2}}}{(2\pi)^2} e^{i \sum_{l=0}^{N} [p^a_{l+\frac{1}{2}}(a_{l+1}-a_l) + p^b_{l+\frac{1}{2}}(b_{l+1}-b_l)]} \qquad (5.53)$$

$$\times e^{i \sum_{l=0}^{N} [p^c_{l+\frac{1}{2}}(c_{l+1}-c_l) - \epsilon\alpha(p^c_{l+\frac{1}{2}} + \frac{(a_{l+1}+a_l)}{2} p^b_{l+\frac{1}{2}} - \frac{(b_{l+1}+b_l)}{2} p^a_{l+\frac{1}{2}} - \frac{(b_{l+1}+b_l)}{2} R)^2]}$$

$$= 2 e^{-iR(a''-a')} \delta[(a''^2 + b''^2) - (a'^2 + b'^2)] \frac{1}{(4\pi i \alpha T)^{\frac{1}{2}}} \sum_{m=-\infty}^{\infty} e^{i \frac{(c''-c'+2\pi m)^2}{4\alpha T}}.$$

The last factor in this result agrees with one obtained by Schulman using different techniques [28]. The δ function tells us that the particle is confined to move on a circle of radius $r = (a'^2 + b'^2)$.

5.3 Propagation with the Universal Propagator

Here we present an argument showing that the universal propagator evolves all wave functions of the system correctly. Let in the following, $U(\alpha, \beta, \gamma, T) = e^{-iT(\alpha X + \beta Y + \gamma J)}$. One can show that

$$U(\alpha, \beta, \gamma, T) = e^{-i[\frac{\alpha}{\gamma} \sin \gamma T + \frac{\beta}{\gamma}(\cos \gamma T - 1)]X} e^{-i[\frac{\beta}{\gamma} \sin \gamma T - \frac{\alpha}{\gamma}(\cos \gamma T - 1)]Y} e^{-i\gamma T J}. \qquad (5.54)$$

Hence one finds, for the time evolution of an arbitrary element of $\mathcal{C} \subset L^2(\mathbf{R}^2 \times \mathbf{S}^1, d\mu(a, b, c))$ under $U(\alpha, \beta, \gamma, T)$ the following
$$\psi_\eta(a, b, c, T) = \langle a, b, c, \eta | U(\alpha, \beta, \gamma, T) | \psi \rangle = e^{-iaR} \times$$

$$\langle \eta | e^{icJ} e^{ibY} e^{iaX} e^{-i[\frac{\alpha}{\gamma} \sin \gamma T + \frac{\beta}{\gamma}(\cos \gamma T - 1)]X} e^{-i[\frac{\beta}{\gamma} \sin \gamma T - \frac{\alpha}{\gamma}(\cos \gamma T - 1)]Y} e^{-i\gamma T J} | \psi \rangle$$

$$= e^{-iaR} \langle \eta | e^{i(c-\gamma T)J} e^{i[b \cos \gamma T - a \sin \gamma T - \frac{\beta}{\gamma} \sin \gamma T - \frac{\alpha}{\gamma}(\cos \gamma T - 1)]Y} \times$$

$$e^{i[a \cos \gamma T + b \sin \gamma T - \frac{\alpha}{\gamma} \sin \gamma T + \frac{\beta}{\gamma}(\cos \gamma T - 1)]X} | \psi \rangle \qquad (5.55)$$

$$= \psi_\eta([a \cos \gamma T + b \sin \gamma T - \frac{\alpha}{\gamma} \sin \gamma T + \frac{\beta}{\gamma}(\cos \gamma T - 1)],$$

$$[b \cos \gamma T - a \sin \gamma T - \frac{\beta}{\gamma} \sin \gamma T - \frac{\alpha}{\gamma}(\cos \gamma T - 1)], [c - \gamma T + 2\pi n]).$$

Where n is an integer such that $-\pi < (c - \gamma T + 2\pi n) \leq \pi$. From our discussion earlier, we know that there exists a universal propagator such that

$$\psi_\eta(a, b, c, T) = \int K(a, b, c, T; a', b', c', 0)\psi_\eta(a', b', c')d\mu(a', b', c'). \tag{5.56}$$

Thus, using $\psi_\eta(a, b, c, T)$ from equation (55) we have

$$\psi_\eta([a\cos\gamma T + b\sin\gamma T - \frac{\alpha}{\gamma}\sin\gamma T + \frac{\beta}{\gamma}(\cos\gamma T - 1)],$$

$$[b\cos\gamma T - a\sin\gamma T - \frac{\beta}{\gamma}\sin\gamma T - \frac{\alpha}{\gamma}(\cos\gamma T - 1)], [c - \gamma T + 2\pi n]) \tag{5.57}$$

$$= \int K(a, b, c, T; a', b', c', 0)\psi_\eta(a', b', c')d\mu(a', b', c').$$

If the above equation is to be valid for arbitrary η and ψ, we must require that

$$K(a, b, c, T; a', b', c', 0) = \frac{2(2\pi)^2}{\overline{R}^2 - R^2}e^{-i\alpha RT} \times$$

$$\delta[a' - a\cos\gamma T - b\sin\gamma T + \frac{\alpha}{\gamma}\sin\gamma T - \frac{\beta}{\gamma}(\cos\gamma T - 1)] \times$$

$$\delta[b' - b\cos\gamma T + a\sin\gamma T + \frac{\beta}{\gamma}\sin\gamma T + \frac{\alpha}{\gamma}(\cos\gamma T - 1)] \times \tag{5.58}$$

$$\sum_{m=-\infty}^{\infty} \{\delta(c' - c + \gamma T - 2\pi m)\}.$$

This expression agrees exactly with our result obtained earlier for the Hamiltonian $H(X, Y, J) = \alpha X + \beta Y + \gamma J$, and therefore establishes its validity.

Now we show that the universal propagator for an arbitrary Hamiltonian evolves the state functions correctly. Let $U(\alpha, \beta, \gamma, 1) = U(\alpha, \beta, \gamma)$. One can represent every bounded function of X, Y, and J as a weak limit of finite linear combinations of the set $U(\alpha, \beta, \gamma)$ [24, 25, 29]. In particular, we can represent every time evolution operator as such a weak limit of finite linear combinations of the $U(\alpha, \beta, \gamma)$. Let us denote by

$$S_n^T = \sum_{j=0}^{n} s_j U(\alpha_j, \beta_j, \gamma_j), \tag{5.59}$$

and also let

$$K_n(a', b', c', T; a, b, c, 0) = \sum_{j=0}^{n} s_j K_{\alpha_j \beta_j \gamma_j}(a', b', c', 1; a, b, c, 0). \tag{5.60}$$

It follows that every time evolution operator is a weak limit of S_n^T, i.e.,

$$\langle \phi | e^{-iTH} | \psi \rangle = \lim_{n \to \infty} \langle \phi | S_n^T | \psi \rangle, \qquad \forall \quad \phi, \psi \in \mathsf{H}. \tag{5.61}$$

Then in particular we have

$$\langle a, b, c, \eta | e^{-iTH} | \psi \rangle = \lim_{n \to \infty} \langle a, b, c, \eta | S_n^T | \psi \rangle, \qquad \forall \quad \psi \in \mathsf{H}. \tag{5.62}$$

Now let us denote by $K(a, b, c, T; a', b', c', 0)$ the universal propagator associated with the time evolution operator e^{-iTH}; Equation (62) can now be written as an equation for linear functionals, and takes the following form,

$$\int K(a', b', c', T; a, b, c, 0) \psi_\eta(a, b, c) d\mu(a, b, c)$$
$$= \lim_{n \to \infty} \int K_n(a', b', c', T; a, b, c, 0) \psi_\eta(a, b, c) d\mu(a, b, c), \tag{5.63}$$

for all $\psi_\eta(a, b, c) \in \mathcal{C}$. Hence, every propagator can be written as the weak* limit of an appropriate K_n, i.e.

$$K(a', b', c', T; a, b, c, 0) = w^\star - \lim_{n \to \infty} K_n(a', b', c', T; a, b, c, 0). \tag{5.64}$$

Although the point is clear from the foregoing, it is worth emphasizing that the universal propagator evolves any state in a way that leaves the choice of the fiducial vector $|\eta\rangle$ invariant.

5.4 Classical Limit

Although the universal propagator has been derived by identifying the relevant Schrödinger equation as one for *three* degrees of freedom, it should nevertheless be true that the classical limit refers to a *single* degree of freedom *constrained to lie on a circle*. This is possible because of the restricted form of the classical and quantum Hamiltonians.

The classical action appropriate to the E(2) coherent state path integral is

$$I_{cl} = \lim_{\hbar \to 0} \int [i\langle a, b, c, \eta | \frac{d}{dt} | a, b, c, \eta \rangle - \langle a, b, c, \eta | H(X, Y, J) | a, b, c, \eta \rangle] dt. \tag{5.65}$$

Usually one restricts the fiducial vector $|\eta\rangle$ such that $\langle\eta|X|\eta\rangle \to R$, as $\overline{R} \to R$, and also that $\langle\eta|Y|\eta\rangle = 0$, $\langle\eta|J|\eta\rangle = 0$. A surface constant fiducial vector that has the above properties is given by

$$\eta_0(r,\theta) = [\frac{2}{(\overline{R}^2 - R^2)}]^{\frac{1}{2}} \frac{e^{Rr\cos\theta}}{(\int\limits_{S^1} e^{2Rr\cos\alpha}d\alpha)^{\frac{1}{2}}} = [\frac{2}{(\overline{R}^2 - R^2)}]^{\frac{1}{2}} \frac{e^{Rr\cos\theta}}{(2\pi I_0(2Rr))^{\frac{1}{2}}}, \quad (5.66)$$

where I_0 is the Bessel function of zeroth order. Hence, under the standard assumptions given above the classical action reads,

$$I_{cl} = \int [\dot{a}R\cos c + \dot{b}R\sin c - H(R\cos c, R\sin c, R(a\sin c - b\cos c))]dt. \quad (5.67)$$

Extremal variation of this expression with respect to a, b, and c holding the end points fixed leads to the equations of motion. The variation with respect to a and b give the same equation, hence the equations of motion are,

$$\dot{c} = H_3,$$

$$-\dot{a}R\sin c + \dot{b}R\cos c = -H_1R\sin c + H_2R\cos c + H_3(aR\cos c + bR\sin c), \quad (5.68)$$

where in the expression above

$$H_i = \frac{\partial H(x_1, x_2, x_3)}{\partial x_i}. \quad (5.69)$$

We note, if we set $p = R(a\sin c - b\cos c)$ that the equations of motion (68) can be written as

$$\dot{c} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial c}, \quad (5.70)$$

which are just Hamilton's equations of motion appropriate to a single degree of freedom (constrained to move on a circle). We denote a generic solution of equation (70) by $p_c(t)$ and $c_c(t)$, i.e.,

$$p(t) = p_c(t) = R[a_c(t)\sin c_c(t) - b_c(t)\cos c_c(t)], \quad c(t) = c_c(t). \quad (5.71)$$

However, when we deal with a more general fiducial vector $|\eta\rangle$, such that

$$\langle\eta|X|\eta\rangle = X_\eta, \quad \langle\eta|Y|\eta\rangle = Y_\eta, \quad \langle\eta|J|\eta\rangle = J_\eta, \tag{5.72}$$

are generally all nonvanishing, the classical action, in the limit of zero dispersion, becomes

$$I_{cl} = \int [\dot{a}(X_\eta \cos c - Y_\eta \sin c) + \dot{b}(Y_\eta \cos c + X_\eta \sin c) + \dot{c}J_\eta$$

$$- H(X_\eta \cos c - Y_\eta \sin c, Y_\eta \cos c + X_\eta \sin c, \tag{5.73}$$

$$J_\eta + X_\eta(a\sin c - b\cos c) + Y_\eta(a\cos c + b\sin c))]dt.$$

In this expression X_η, Y_η, and J_η are time independent constants. The term $\int(\dot{a}R + \dot{c}J_\eta)dt = (a'' - a')R + (c'' - c')J_\eta$ is a pure surface term and will not affect the equations of motion; it could be eliminated simply by a phase change of the coherent states. Extremal variation holding the end points fixed leads to the equations of motion

$$\dot{c} = H_3,$$

$$\dot{a}(X_\eta \sin c + Y_\eta \cos c) + \dot{b}(Y_\eta \sin c - X_\eta \cos c)$$

$$+ H_3(aX_\eta \cos c + bX_\eta \sin c - aY_\eta \sin c + bY_\eta \cos c) \tag{5.74}$$

$$= H_1(X_\eta \sin c + Y_\eta \cos c) + H_2(Y_\eta \sin c - X_\eta \cos c).$$

Now, if we set $p = X_\eta(a\sin c - b\cos c) + Y_\eta(a\cos c + b\sin c)$ the equations of motion can be written as

$$\dot{c} = \frac{\partial H(X_\eta \cos c - Y_\eta \sin c, Y_\eta \cos c + X_\eta \sin c, p + J_\eta)}{\partial p},$$

$$\dot{p} = -\frac{\partial H}{\partial c}. \tag{5.75}$$

Which have as their solutions

$$c(t) = c_c(t),$$

$$p_c(t) = X_\eta(a_c(t)\sin c_c(t) - b_c(t)\cos c_c(t)) \tag{5.76}$$

$$+ Y_\eta(a_c(t)\cos c_c(t) + b_c(t)\sin c_c(t)) - J_\eta.$$

Observe that the generally nonvanishing values of X_η, Y_η, and J_η are vestiges of the coherent-state representation induced by the fiducial vector $|\eta\rangle$ that remain even in the limit $\hbar \to 0$.

5.4.1 Classical Solutions

In the case of the universal propagator the expression that serves as the classical action is identified as

$$I_{cl} = \int [p^a \dot{a} + p^b \dot{b} + p^c \dot{c} - H(p^a + R, p^b, p^c + ap^b - bp^a - bR)]dt \qquad (5.77)$$

Extremal variation of this expression holding the end points fixed leads to the following set of equations

$$\dot{a} = H_1 - bH_3, \quad \dot{b} = H_2 + aH_3, \quad \dot{c} = H_3,$$
$$\dot{p^a} = -p^b H_3, \quad \dot{p^b} = (p^a + R)H_3, \quad \dot{p^c} = 0. \qquad (5.78)$$

The solution to the last three equations can be written as

$$p^a(t) = \alpha_1 \cos c(t) - \alpha_2 \sin c(t) - R,$$

$$p^b(t) = \alpha_2 \cos c(t) + \alpha_1 \sin c(t), \qquad (5.79)$$

$$p^c(t) = \alpha_3,$$

where α_1, α_2, and α_3 are arbitrary constants of integration. The first two equations can be combined to give

$$\frac{d}{dt}[ap^b - b(p^a + R)] = H_1 p^b - H_2(p^a + R) = -\frac{\partial H}{\partial c}. \qquad (5.80)$$

Hence we see that if we define $p = ap^b - b(p^a + R)$, the equations of motion can be written as

$$\dot{p} = -\frac{\partial}{\partial c}H(\alpha_1 \cos c - \alpha_2 \sin c, \alpha_2 \cos c + \alpha_1 \sin c, p + \alpha_3),$$
$$\dot{c} = \frac{\partial}{\partial p}H(\alpha_1 \cos c - \alpha_2 \sin c, \alpha_2 \cos c + \alpha_1 \sin c, p + \alpha_3), \qquad (5.81)$$

which are seen to be Hamilton's equations for a single degree of freedom. The solution to the above equations can be written as

$$p(t) = p_c(t) = a_c(t)[\alpha_2 \cos c_c(t) + \alpha_1 \sin c_c(t)]$$

$$- b_c(t)[\alpha_1 \cos c_c(t) - \alpha_2 \sin c_c(t)] - \alpha_3, \qquad (5.82)$$

$$c(t) = c_c(t).$$

Where α_1, α_2, and α_3 are arbitrary constants as mentioned before. Among all possible values of α_1, α_2, and α_3 are those that coincide with X_η, Y_η, J_η corresponding to the fiducial vector $|\eta\rangle$. Thus, we find that the set of classical solutions obtained from the action appropriate to the universal propagator includes every possible solution of the classical equations of motion appropriate to the most general coherent-state propagator.

The techniques discussed in this chapter may be extended to $E(n)$ coherent states, suitable for quantizing a system whose configuration space is the S^{n-1} sphere [27].

CHAPTER 6
DISCUSSION AND CONCLUSIONS

We will now summarize the results of this thesis. Our goal had been to study methods of handling constrained systems in a functional integral framework. Dynamical constraints are one of two types; first-class or second-class constraints. We have studied three models, each with a finite number of degrees of freedom, and each with either first-class constraints or second-class constraints. Our 'tools' have been coherent-states, the formalism surrounding which is exceptionally rich.

The principal assertion of this thesis is that, for constrained systems, in the construction of the path integral representation of a propagator one should use a projection operator rather than the resolution of unity at every time slice. The projection operator, for a given system, is constructed using a complete set of states in the physical subspace of the system. The main difference in the treatment of systems with either first-class or second-class constraints is the way in which the physical subspace is identified. Once the physical subspace of the system is identified and a complete set of states in this subspace found it is straightforward to construct the desired projection operator. The projection operator is used at each time slice of the propagator and as a state of the system evolves in time, the projection operator ensure that the state remains in the physical subspace at every infinitesimal time step of the evolution.

In chapter 3 we studied a model with two first-class constraints. We constructed the projection operator appropriate to the model. It is found that the projection operator can be written as a properly weighted integral over independent bras and kets which were called *Bicoherent States*. The use of the projection operator leads to the correct measure for the path integral. Also, the projection operator which is an integral over bicoherent

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states leads to an action that is complex and has twice as many labels as the standard action. We found that in the classical limit among these labels, which number twice the normal labels, pairs of them follow identical trajectories which correspond to the classical trajectories. It is also found that on the classical trajectories the real part of the action reduces to the standard classical action while the imaginary part becomes just a surface term. The projection operator leads to a measure that is path dependent. This path dependent measure is 'modulated' by the imaginary part of the action

The formalism studied here additionally has the desirable feature whereby one can turn on and off the constraints as needed. For instance, for the model studied in chapter three if we wish the system to evolve under the constrained Hamiltonian $\mathcal{H}_1 = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{x}) + y\mathbf{p}T\mathbf{x} + u\mathbf{p}T\mathbf{x} + v\pi$ for an interval of time T_1 with the two first-class constraints, and subsequently to evolve under an unconstrained Hamiltonian say $\mathcal{H}_2 = \frac{1}{2}\mathbf{p}^2 + V(\mathbf{x})$ for a period T_2 the propagator is given by

$$\langle \alpha'', \beta''; (T_1 + T_2)|\alpha, \beta; 0\rangle = \int \frac{d^2\alpha' d^2\beta'}{\pi^2} \langle \alpha'', \beta''|e^{-iT_2\mathcal{H}_2}|\alpha', \beta'\rangle \delta(\alpha'^\star\beta' - \beta'^\star\alpha')$$
$$\times \langle \alpha', \beta', \frac{c_1 T + c_5}{\sqrt{2}}, \frac{c_3 + ic_4}{\sqrt{2}}, \frac{c_1 + ic_2}{\sqrt{2}}|e^{-iT_1\mathcal{H}_1}|\alpha, \beta, \frac{c_5}{\sqrt{2}}, \frac{c_3 + ic_4}{\sqrt{2}}, \frac{c_1 + ic_2}{\sqrt{2}}\rangle, \tag{6.1}$$

where the factor $\langle \alpha'', \beta''|e^{-iT_2\mathcal{H}_2}|\alpha', \beta'\rangle$ in the integrand above is evaluated in the usual manner by introducing resolutions of unity at each time slice, whereas the term $\langle \alpha', \beta', \frac{c_1T+c_5}{\sqrt{2}}, \frac{c_3+ic_4}{\sqrt{2}}, \frac{c_1+ic_2}{\sqrt{2}}|e^{-iT_1\mathcal{H}_1}|\alpha, \beta, \frac{c_5}{\sqrt{2}}, \frac{c_3+ic_4}{\sqrt{2}}, \frac{c_1+ic_2}{\sqrt{2}}\rangle$ would be evaluated as discussed in section 3.3. The Dirac delta function in the integrand above ensures that at time $t = T_1$ the system is on the constraint surface.

In chapter 4 we apply the formalism of bicoherent states to systems with second-class constraints. In this chapter we have studied two models. The crucial step of the quantization process is the identification of the physical subspace. For systems with second-class constraints one starts of by identifying $\mathcal{H}_{ph}$, on the way to identifying a complete set of vectors in the physical subspace. For the two models studied in this

chapter the physical subspaces were spanned by the eigenstates of $\mathcal{H}_{ph}$ which were known. We used these states to form the projection operator which was used in the construction of the path integral representation of the propagator. So, in essence we wrote the path integrals for problems whose solution we already knew. But in the case where one does not know the eigenstates of $\mathcal{H}_{ph}$, all one has to do is find a suitable operator, say $\mathcal{O}$, that commutes with $\mathcal{H}_{ph}$ and whose eigenstates are known. The eigenstates of $\mathcal{O}$ then span the physical subspace and can be used to construct the required projection operator and hence a path integral for the propagator.

In conclusion, we note bicoherent states are quite versatile. In appendix A we saw that they can be used to construct path integrals for unconstrained systems. Thus, the quantization 'recipe' discussed here using bicoherent states amounts to obtaining the correct measure for the path integrals. So, one can have, in principle, an identical action in the path integral for a constrained and an unconstrained system but the measure would be entirely different. Also, all integrals are regular when bicoherent states are used just as in the case when coherent states are used. The formalism developed here and applied to systems with a finite number of degrees of freedom should be extendable to field theories.

APPENDIX A
BICOHERENT STATES AND PATH INTEGRALS

In this section we highlight the main features of path integrals constructed using bicoherent states. For simplicity we consider a system without constraints and with a single degree of freedom. To begin with, notice that the unit operator can be written as

$$
\begin{aligned}
1 = 1^2 &= \int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| \int \frac{d^2\beta}{\pi}|\beta\rangle\langle\beta| \\
&= \int \frac{d^2\alpha d^2\beta}{\pi^2}|\alpha\rangle\langle\beta|e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha^\star\beta},
\end{aligned}
\tag{A.1}
$$

a weighted integral over bicoherent states. We will use this form of unity in our construction of the path integral representation of the propagator. Consider the Hamiltonian $H = p^2/2 + V(x)$. In the construction of the propagator we use the normal ordered Hamiltonian $\mathcal{H} = : H :$. Thus, the propagator is given by

$$
\begin{aligned}
\langle\alpha''|e^{-iT\mathcal{H}}|\alpha'\rangle &= \int \langle\alpha''|e^{-i\epsilon\mathcal{H}}|\alpha_N\rangle\langle\beta_N|e^{-i\epsilon\mathcal{H}}|\alpha_{N-1}\rangle...\langle\beta_1|e^{-i\epsilon\mathcal{H}}|\alpha'\rangle \prod_{n=1}^{N} d\mu_n'' \\
&= \int \prod_{n=0}^{N}\langle\beta_{n+1}|e^{-i\epsilon\mathcal{H}}|\alpha_n\rangle \prod_{n=1}^{N} d\mu_n'',
\end{aligned}
\tag{A.2}
$$

where $(N+1)\epsilon = T$ and $n = 1, 2...N$. In the equation above, the boundary conditions are $(\alpha_0, \beta_{N+1}) = (\alpha', \alpha'')$ and the measure at each time slice is given by

$$
d\mu_n'' = \frac{d^2\alpha_n d^2\beta_n}{\pi^2}e^{-\frac{1}{2}(|\alpha_n|^2+|\beta_n|^2)+\alpha_n^\star\beta_n}.
\tag{A.3}
$$

For small ϵ, we evaluate, to order ϵ, each term in the integrand in (A.2) as follows:

$$
\begin{aligned}
\langle\beta_{n+1}|e^{-i\epsilon\mathcal{H}}|\alpha_n\rangle &\simeq \langle\beta_{n+1}|[1 - i\epsilon\mathcal{H}]|\alpha_n\rangle, \\
&= \langle\beta_{n+1}|\alpha_n\rangle[1 - i\epsilon H_{n+1,n}] \simeq \langle\beta_{n+1}|\alpha_n\rangle e^{-i\epsilon H_{n+1,n}},
\end{aligned}
\tag{A.4}
$$

where $H_{n+1,n}$ in the expression above is

$$
H_{n+1,n} = \frac{\langle\beta_{n+1}|\mathcal{H}|\alpha_n\rangle}{\langle\beta_{n+1}|\alpha_n\rangle}.
\tag{A.5}
$$

Also, the overlap of coherent-states at each time slice is

$$\langle \beta_{n+1} | \alpha_n \rangle = e^{-\frac{1}{2}(|\beta_{n+1}|^2 + |\alpha_n|^2) + \beta_{n+1}^\star \alpha_n}. \tag{A.6}$$

Thus, provided the integrals exist, the propagator is given by

$$\langle \alpha'' | e^{-iT\mathcal{H}} | \alpha' \rangle = \int \prod_{n=0}^{N} e^{-\frac{1}{2}(|\beta_{n+1}|^2 + |\alpha_n|^2) + \beta_{n+1}^\star \alpha_n - i\epsilon H_{n+1,n}} \prod_{n=1}^{N} d\mu_n''. \tag{A.7}$$

In the expression above we notice that the factor $e^{-\frac{1}{2}(|\beta_{n+1}|^2 + |\alpha_n|^2)}$, except at the end points, can be absorbed in the measure. So, our propagator becomes

$$\langle \alpha'' | e^{-iT\mathcal{H}} | \alpha' \rangle = e^{-\frac{1}{2}(|\alpha''|^2 + |\alpha'|^2)} \int e^{\sum_{n=0}^{N} [\beta_{n+1}^\star \alpha_n - i\epsilon H_{n+1,n}]} \prod_{n=1}^{N} d\mu_n', \tag{A.8}$$

where the measure at each time slice has changed slightly and is now given by

$$d\mu_n' = \frac{d^2\alpha_n d^2\beta_n}{\pi^2} e^{-(|\alpha_n|^2 + |\beta_n|^2) + \alpha_n^\star \beta_n}. \tag{A.9}$$

In our quest to express the right hand side of (A.8) as a path integral, we rewrite part of the exponent as follows,

$$\sum_{n=0}^{N} \beta_{n+1}^\star \alpha_n = \sum_{n=0}^{N} \frac{1}{2}[(\beta_{n+1}^\star - \beta_n^\star)\alpha_n - \beta_{n+1}^\star (\alpha_{n+1} - \alpha_n)]$$
$$+ \sum_{n=0}^{N} \frac{1}{2}[\beta_n^\star \alpha_n + \beta_{n+1}^\star \alpha_{n+1}]. \tag{A.10}$$

Notice that in the above equation the terms (β_0, α_{N+1}) have not been defined yet. The factors containing these terms cancel and so they can take on arbitrary values. We assign them the following values: $(\beta_0, \alpha_{N+1}) = (\alpha', \alpha'')$. Also, the second sum in (A.10) can be absorbed in the measure and so our propagator is finally written as

$$\langle \alpha'' | e^{-iT\mathcal{H}} | \alpha' \rangle = \int e^{\sum_{n=0}^{N} \{\frac{1}{2}[(\beta_{n+1}^\star - \beta_n^\star)\alpha_n - \beta_{n+1}^\star (\alpha_{n+1} - \alpha_n)] - i\epsilon H_{n+1,n}\}} \prod_{n=1}^{N} d\mu_n, \tag{A.11}$$

where the measure at each slice is now

$$d\mu_n = \frac{d^2\alpha_n d^2\beta_n}{\pi^2} e^{-(|\alpha_n|^2 + |\beta_n|^2) + \alpha_n^\star \beta_n + \beta_n^\star \alpha_n} = \frac{d^2\alpha_n d^2\beta_n}{\pi^2} e^{-|\alpha_n - \beta_n^\star|^2}. \tag{A.12}$$

Thus, in the limit $N \to \infty$, $\epsilon \to 0$ with $(N + 1)\epsilon = T$, the right side of (A.11) can formally be written as an integral over continuous and differentiable paths as

$$\langle \alpha'' | e^{-iT\mathcal{H}} | \alpha' \rangle = \int e^{i \int \{\frac{i}{2}(\beta^\star \dot{\alpha} - \dot{\beta}^\star \alpha) - \langle \mathcal{H} \rangle\} dt} \mathcal{D}\mu, \tag{A.13}$$

where

$$\langle \mathcal{H} \rangle = \frac{\langle \beta | \mathcal{H} | \alpha \rangle}{\langle \beta | \alpha \rangle} \tag{A.14}$$

and the discrete form of the measure is

$$\mathcal{D}\mu = \prod_n \{\frac{d^2 \alpha_n d^2 \beta_n}{\pi^2} e^{-(|\alpha_n|^2 + |\beta_n|^2) + \alpha_n^\star \beta_n + \beta_n^\star \alpha_n}\}. \tag{A.15}$$

Notice that the phase-space action

$$S = \int \{\frac{i}{2}(\beta^\star \dot{\alpha} - \dot{\beta}^\star \alpha) - \langle \mathcal{H} \rangle\} dt \tag{A.16}$$

obtained from the path integral representation of the propagator constructed using bicoherent states is complex.

In the case of the harmonic oscillator $\mathcal{H} = a^\dagger a$, and the action is

$$S = \int \{\frac{i}{2}(\beta^\star \dot{\alpha} - \dot{\beta}^\star \alpha) - \beta^\star \alpha\} dt. \tag{A.17}$$

Defining $\alpha = (q_1 + ip_1)/\sqrt{2}$ and $\beta = (q_2 + ip_2)/\sqrt{2}$ and inserting these definitions in the action above we obtain

$$S = \int \{\frac{1}{4}(p_2 \dot{q}_1 - q_2 \dot{p}_1 + p_1 \dot{q}_2 - q_1 \dot{p}_2) - \frac{1}{2}(q_2 q_1 + p_2 p_1)\} dt$$
$$+ i \int \{\frac{1}{4}(q_2 \dot{q}_1 + p_2 \dot{p}_1 - q_1 \dot{q}_2 - p_1 \dot{p}_2) - (q_2 p_1 - p_2 q_1)\} dt. \tag{A.18}$$

In this thesis we identify the real part of the action obtained from the bicoherent state construction, with the standard classical phase space action. We see that for the example of the harmonic oscillator, the action, obtained from our construction using bicoherent states, has twice as many labels as the usual action. This doubling of labels and the fact that the action is complex are two general features of path integrals constructed using bicoherent states.

APPENDIX B
THE (a, b, c) MODEL

In this appendix we show that the (a, b, c) model studied in chapter 5 can be looked upon as a system having second-class constraints. Here we closely follow the notes on the subject by Whiting [11]. The model deals with a situation in which the Hamiltonian has an odd number of arguments; a situation that arises frequently for coherent states based on generalized group representations. Thus, the function

$$F(a, b, c) = V(c) + \frac{1}{2}(a^2 + b^2) \tag{B.1}$$

is strictly speaking, not a Hamiltonian since there is no pairing of variables. The interest in this problem arose when the author of the model was studying the Lagrangian

$$\mathcal{L} = \dot{a}\cos c + \dot{b}\sin c - V(c) - \frac{1}{2}(a^2 + b^2). \tag{B.2}$$

It is obvious that if we now try to identify $p_a = \cos c$, $p_b = \sin c$, and $p_c = 0$ then these identities must be thought of as giving rise to the constraints

$$\phi_0 = p_a - \cos c, \qquad \phi_1 = p_b - \sin c, \qquad \phi_2 = p_c. \tag{B.3}$$

Thus, the primary Hamiltonian becomes

$$\mathcal{H} = V(c) + \frac{1}{2}(a^2 + b^2) + \lambda_0(p_a - \cos c) + \lambda_1(p_b - \sin c) + \lambda_2 p_c. \tag{B.4}$$

As usual we look for secondary constraints and find

$$\dot{\phi}_0 = \{\phi_0, \mathcal{H}\} = -a + \lambda_2 \sin c, \qquad \dot{\phi}_1 = \{\phi_1, \mathcal{H}\} = -b - \lambda_2 \cos c,$$

$$\dot{\phi}_2 = \{\phi_2, \mathcal{H}\} = -[\frac{\partial V}{\partial c} + \lambda_0 \sin c - \lambda_1 \cos c], \tag{B.5}$$

which leads to the additional constraint $\psi = a \cos c + b \sin c$. The requirement that ψ hold at time leads to the final constraint

$$\dot{\psi} = \lambda_0 \cos c + \lambda_1 \sin c + \lambda_2(-a \sin c + b \cos c). \tag{B.6}$$

Now we have enough equations to determine all the λ's. Once all the λ's are found, all the constraints turn out to be second-class.
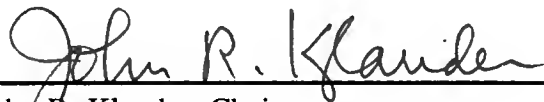
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BIOGRAPHICAL SKETCH

The author was born in Cochin, India, on the 12th of September 1962. He spent the first, happy, 24 years of his life in India. He earned a Bachelor of Science, with physics, chemistry, and mathematics as the main subjects. He taught himself calculus and most of the other math during his bachelors. He also earned his Master of Science in physics in India in the summer of 1984. He came to the United States in the fall of 1986 to pursue a doctorate in physics. After a few, long, faltering steps he is finally graduating. He met Jocelyn, to whom he is married, here in Gainesville in the fall of 1993. He has had a good time in the United States. He has enjoyed his studies in physics and also likes the biological sciences quite a bit. He likes sports and reading among other things.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.
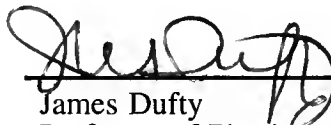
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Professor of Physics

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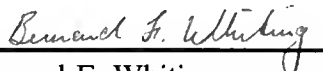
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December 1995

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